# A rigorous approach to the magnetic response in disordered systems

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**Abstract:**

This paper is a part of an ongoing study on the diamagnetic behavior of a 3-dimensional quantum gas of non-interacting charged particles subjected to an external uniform magnetic field together with a random electric potential. We prove the existence of an almost-sure non-random thermodynamic limit for the grand-canonical pressure, magnetization and zero-field orbital magnetic susceptibility. We also give an explicit formulation of these thermodynamic limits. Our results cover a wide class of physically relevant random potentials which model not only *crystalline disordered solids*, but also *amorphous solids*.

PACS-2010 number: 05.30.-d, 51.30.+i, 51.60.+a, 02.30.Tb, 75.20.En, 75.20.Ck

MSC-2010 number: 82B10, 82B44, 81Q10, 81V99.

Keywords: Operator theory, Random Scrhrödinger operators, quantum statistical mechanics, diamagnetism, disordered solids.

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## 1. Introduction and main results

1.1. **Introduction.** In this paper we study rigorously the diamagnetic response of a quantum gas of non-interacting charged particles trapped in an amorphous medium and subjected to an uniform magnetic field of intensity $B \geq 0$. Under the grand-canonical conditions and in the weak-field regime, this response is completely characterized by the pressure and its first two derivatives w.r.t. $B$, i.e. the magnetization and the susceptibility. Especially we are focusing on the bulk response which is of great interest since it is independent of the boundary effects. This is obtained by proving the existence of the thermodynamic limit for these quantities firstly defined at finite volume. From a mathematical point of view, this consists of showing that the derivatives of the pressure w.r.t. $B$ (performed at finite volume) commute with the thermodynamic limit.

Our paper is an extension of the works of Briet *et al.* [3, 4, 5] where the case of a perfect quantum gas has been treated. All these papers are in fact in the continuation of a study initiated by Angelescu *et al.* [1, 2]; for a brief review see also [10, 4]. In the regime of positive temperature $T$ and small fugacity $z$, it is proved in [3, 4] the existence of the thermodynamic limit for the pressure and all its derivatives w.r.t. $B$ for any positive value of the cyclotron frequency, $b := qB/c$. This proof is based on a main ingredient: the application of the so-called gauge invariant magnetic perturbation theory to the corresponding Gibbs semigroups, see also [10]. Afterwards in [5], they extend these results to the $z$-complex domain of analyticity of the pressure, through the Vitali convergence theorem. Consequently they obtained the existence but no explicit formula of the limits for all admissible values of $z$.

Recently, all these results have been improved in [6] covering the case of periodic interactions with local singularities; basically of the Kato class. But in [6], only the limit of the pressure has been considered. The proof is essentially based on the Pastur-Shubin formula for the integrated density of states [15]. Later on the generalized susceptibilities were studied in [22]. These results have been used in [7] to get a zero-field orbital susceptibility formula for a Bloch electron gas and the justification of the Landau-Peierls approximation at small density and zero temperature.

In this paper, the background electric potential is assumed to be a $\mathbb{G}^3$-ergodic ($\mathbb{G} = \mathbb{Z}$ or $\mathbb{R}$) random field having two types of singularities: local singularities and a polynomial growth at infinity, see assumptions (R1)-(R2) below. These assumptions cover most of the electric potentials widely used in the quantum theory of solids, see Section 1.3 for examples. Our main results prove generically the existence of an almost-sure non-random thermodynamic limit for the pressure, magnetization and zero-field orbital susceptibility. Furthermore we give an explicit expression of these limits on the maximal (independent of $b$) $z$-complex domains without resorting to the Vitali theorem. These significant advances are made possible by employing the gauge invariant magnetic perturbation theory to control the perturbed resolvent operator, see [11] for further applications.

1.2. **The setting and the main result.** Consider a 3-dimensional quantum gas composed of non-relativistic identical charged particles, obeying either the Bose-Einstein or the Fermi-Dirac statistics, and subjected to an external constant magnetic field. Since we are only interested in orbital diamagnetic effects, we do not consider the spin of particles. Besides each particle interacts with a random electric potential (the sense will make precise hereafter) modeling a disordered medium. The interactions between particles are neglected (strongly diluted gas assumption) and the gas is at thermal equilibrium.

Let us precise our assumptions. The gas is confined in a cubic box centered at the origin

given by $\Lambda_L = \Lambda_L(\mathbf{0}) := (-L/2, L/2)^3$, $L > 0$. The magnetic field is defined by $\mathbf{B} := (0, 0, B)$ with $B \geq 0$, and we use the symmetric gauge, i.e. the magnetic vector potential is defined by $B\mathbf{a}(\mathbf{x}) := \frac{B}{2}\mathbf{e}_3 \wedge \mathbf{x} = \frac{B}{2}(-x_2, x_1, 0)$. In the following we denote $b := \frac{qB}{c} \in \mathbb{R}$.
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and $\mathbb{E}[\cdot] := \int_\Omega \mathbb{P}(\mathrm{d}\omega)(\cdot)$ be the associated expectation. We consider random electric potentials, i.e. scalar random fields $V : \Omega \times \mathbb{R}^3 \to \mathbb{R}$, $(\omega, \mathbf{x}) \mapsto V^{(\omega)}(\mathbf{x})$ which are assumed to be jointly measurable with respect to the product of the $\sigma$-algebra $\mathcal{F}$ on $\Omega$ and the Borel-algebra $\mathcal{B}(\mathbb{R}^3)$. In the whole of the paper we suppose:

(E) $V^{(\omega)}$ is a $\mathbb{R}^3$-ergodic random field.

Recall that this assumption (see e.g. [17]) requires the existence of an ergodic group $\{\tau_\mathbf{k}\}_{\mathbf{k}\in\mathbb{R}^3}$ of measure-preserving automorphisms on $\Omega$ s.t. $V^{(\omega)}$ is $\mathbb{R}^3$-stationary in the sense that:

$$V^{(\tau_\mathbf{k}\omega)}(\mathbf{x}) = V^{(\omega)}(\mathbf{x} - \mathbf{k}) \quad \forall \mathbf{x} \in \mathbb{R}^3,\ \forall \mathbf{k} \in \mathbb{R}^3,\ \forall \omega \in \Omega. \tag{1.1}$$

(R) The realizations of $V^{(\omega)}$ are given by:

$$V^{(\omega)}(\mathbf{x}) = V_1^{(\omega)}(\mathbf{x}) + V_2^{(\omega)}(\mathbf{x}) \quad \mathbf{x} \in \mathbb{R}^3,\ \omega \in \Omega, \tag{1.2}$$

where $\mathbb{P}$-a.s. on $\Omega$:

(R1) $V_1^{(\omega)}$ is an uniformly locally integrable function, i.e. $V_1^{(\omega)} \in L^p_{\mathrm{uloc}}(\mathbb{R}^3)$ with $p > 3$.

(R2) $V_2^{(\omega)}$ obeys the conditions:

$$0 \leq V_2^{(\omega)}(\mathbf{x}) \leq c_\alpha(\omega)(1 + |\mathbf{x}|^\alpha) \quad \text{with } \alpha \in (0, \frac{1}{3}) \text{ and } c_\alpha(\omega) > 0. \tag{1.3}$$

Recall that the space $L^p_{\mathrm{uloc}}(\mathbb{R}^3)$ consists of measurable functions $f : \mathbb{R}^3 \to \mathbb{C}$ satisfying:

$$\|f\|_{1\leq p<\infty,\mathrm{uloc}} := \sup_{\mathbf{x}\in\mathbb{R}^3} \left( \int_{|\mathbf{x}-\mathbf{y}|<1} \mathrm{d}\mathbf{y}\, |f(\mathbf{y})|^p \right)^{\frac{1}{p}} < \infty, \quad \|f\|_{\infty,\mathrm{uloc}} := \operatorname{ess\,sup}_{\mathbf{x}\in\mathbb{R}^3} |f(\mathbf{x})| < \infty. \tag{1.4}$$

We discuss below about the choice of these assumptions, see Section 1.3.

Introduce the 'one-particle' Hamiltonian in $L^2(\Lambda_L)$. On $\mathcal{C}_0^\infty(\Lambda_L)$ consider the operator:

$$H_L(b, \omega) := \frac{1}{2}(-i\nabla - b\mathbf{a})^2 + V^{(\omega)}, \quad b \in \mathbb{R}. \tag{1.5}$$

It is well-known (see e.g. [14, Prop. 2.1]) that $\mathbb{P}$-a.s. on $\Omega$, $\forall b \in \mathbb{R}$, (1.5) defines a family of self-adjoint and bounded below operators for any $L \in (0, \infty)$, denoted again by $H_L(b, \omega)$, with domain $D(H_L(b, \omega)) = \mathcal{H}_0^1(\Lambda_L) \cap \mathcal{H}^2(\Lambda_L)$. Obviously this definition corresponds to choose Dirichlet boundary conditions on $\partial\Lambda_L$. Moreover $H_L(b, \omega)$ has purely discrete spectrum; we denote the set of eigenvalues (counting multiplicities and in increasing order) by $\{e_j^{(\omega)}(b, L)\}_{j\geq 1}$. Besides by [6, Prop. 2.2], $\mathbb{P}$-a.s. on $\Omega$, $\{H_L(b, \omega), b \in \mathbb{C}\}$ is a type (A)-entire family of operators.

When $L = \infty$, define on $\mathcal{C}_0^\infty(\mathbb{R}^3)$ the operator:

$$H_\infty(b, \omega) := \frac{1}{2}(-i\nabla - b\mathbf{a})^2 + V^{(\omega)}, \quad b \in \mathbb{R}. \tag{1.6}$$

Then $\mathbb{P}$-a.s. on $\Omega$, $\forall b \in \mathbb{R}$, $H_\infty(b, \omega)$ is essentially self-adjoint and its self-adjoint extension is bounded below, see [23, Thm. B.13.4]. Furthermore $H_\infty(b, \omega)$ is a family of $\mathbb{R}^3$-ergodic self-adjoint operators. This comes from the measurability of the mapping $\omega \in \Omega \mapsto H_\infty(b, \omega)$, see [19, Coro. 3], associated to the assumption (E) which leads to the covariance relation $T_{\mathbf{k},b} H_\infty(b, \omega) T_{-\mathbf{k},b} = H_\infty(b, \tau_\mathbf{k}\omega)$, $\forall \mathbf{k} \in \mathbb{R}^3$. Here $\{T_{\mathbf{k},b}\}_{\mathbf{k}\in\mathbb{R}^3}$ stands for the family of the usual real magnetic translations, see (5.6). We denote by $\Sigma$ the $\mathbb{P}$-a.s. spectrum of $H_\infty(0, \omega)$, [21].

Our analysis is based on the fact that due to assumption (R), the variational principle and the diamagnetic inequality [23], imply:

$$\forall\, b \in \mathbb{R}, \quad \inf \sigma(H_L(b,\omega)) \geq \inf \sigma(H_\infty(b,\omega)) \geq E_0, \quad E_0 := \inf \Sigma, \tag{1.7}$$

as soon as the corresponding self-adjoint operators are well-defined.

Let us recall the basic points of the grand-canonical formalism of the quantum statistical mechanics. Let $\beta := (k_B T)^{-1} > 0$ be the 'inverse' temperature ($k_B$ is the Boltzmann constant). Define the domains $\mathcal{D}_\epsilon = \mathcal{D}_\epsilon(E_0)$, $\epsilon = \pm 1$, by:

$$\mathcal{D}_{-1} := \mathbb{C} \setminus [\mathrm{e}^{\beta E_0}, +\infty), \quad \mathcal{D}_{+1} := \mathbb{C} \setminus (-\infty, -\mathrm{e}^{\beta E_0}]. \tag{1.8}$$

In the following the parameter $\epsilon = -1$ refers to the bosonic case, $\epsilon = +1$ to the fermionic case. For $\beta > 0$, $b \in \mathbb{R}$ and $z \in \mathcal{D}_\epsilon \cap \mathbb{R}_+^*$, the finite-volume pressure and density are defined as [1, 2]:

$$P_L^{(\omega)}(\beta, b, z, \epsilon) := \frac{\epsilon}{\beta|\Lambda_L|} \mathrm{Tr}_{L^2(\Lambda_L)} \ln(\mathbb{I} + \epsilon z \mathrm{e}^{-\beta H_L(b,\omega)}) = \frac{\epsilon}{\beta|\Lambda_L|} \sum_{j=1}^{\infty} \ln(1 + \epsilon z \mathrm{e}^{-\beta e_j^{(\omega)}(b,L)}), \tag{1.9}$$

$$\rho_L^{(\omega)}(\beta, b, z, \epsilon) := \beta z \frac{\partial P_L^{(\omega)}}{\partial z}(\beta, b, z, \epsilon) = \frac{1}{|\Lambda_L|} \sum_{j=1}^{\infty} \frac{z \mathrm{e}^{-\beta e_j^{(\omega)}(b,L)}}{1 + \epsilon z \mathrm{e}^{-\beta e_j^{(\omega)}(b,L)}}. \tag{1.10}$$

The relations (1.9)-(1.10) are well-defined since $\mathbb{P}$-a.s. on $\Omega$, $\forall b \in \mathbb{R}$, the semigroup $\{\mathrm{e}^{-\beta H_L(b,\omega)}, \beta > 0\}$ is trace class, see [6, Eq. (2.12)]. Moreover from [6, Thm. 1.1], then $\mathbb{P}$-a.s. on $\Omega$, $\forall \beta > 0$, $P_L^{(\omega)}(\beta, \cdot, \cdot, \epsilon)$ is an analytic function in $(z, b) \in \mathcal{D}_\epsilon \times \mathbb{R}$. This allows us to define the finite-volume magnetization and orbital susceptibility at $\beta > 0$, $b \in \mathbb{R}$ and $z \in \mathcal{D}_\epsilon \cap \mathbb{R}_+^*$ as [1, 3, 4]:

$$\mathcal{X}_{L,n}^{(\omega)}(\beta, b, z, \epsilon) := \left(\frac{q}{c}\right)^n \frac{\partial^n P_L^{(\omega)}}{\partial b^n}(\beta, b, z, \epsilon) \quad n = 1, 2. \tag{1.11}$$

Hereafter we will sometimes use the notation $\mathcal{X}_{L,0}^{(\omega)}(\beta, b, z, \epsilon) = P_L(\beta, b, z, \epsilon)$.

We now want to formulate our main results. We first introduce some notations.
By [23, Thm. B.7.2], $\mathbb{P}$-a.s. on $\Omega$, $\forall b \in \mathbb{R}$ and $\forall \xi \in \mathbb{C} \setminus [E_0, \infty)$, $R_\infty(b, \omega, \xi) := (H_\infty(b,\omega) - \xi)^{-1}$ has an integral kernel $R_\infty^{(1)}(\cdot\,, \cdot\,; b, \omega, \xi)$ jointly continuous on $\mathbb{R}^6 \setminus D_\infty$, where $D_\infty := \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^6 : \mathbf{x} = \mathbf{y}\}$. Under the same conditions, let $T_{j,\infty}(b, \omega, \xi)$, $j = 1, 2$, be the operators on $L^2(\mathbb{R}^3)$ defined via their integral kernel:

$$T_{1,\infty}(\mathbf{x}, \mathbf{y}; b, \omega, \xi) := \mathbf{a}(\mathbf{x} - \mathbf{y}) \cdot (i\nabla_{\mathbf{x}} + b\mathbf{a}(\mathbf{x})) R_\infty^{(1)}(\mathbf{x}, \mathbf{y}; b, \omega, \xi), \tag{1.12}$$

$$T_{2,\infty}(\mathbf{x}, \mathbf{y}; b, \omega, \xi) := \frac{1}{2} \mathbf{a}^2(\mathbf{x} - \mathbf{y}) R_\infty^{(1)}(\mathbf{x}, \mathbf{y}; b, \omega, \xi), \quad (\mathbf{x}, \mathbf{y}) \in \mathbb{R}^6 \setminus D_\infty. \tag{1.13}$$

Let $\beta > 0$, $b \in \mathbb{R}$, $z \in \mathcal{D}_\epsilon$, and $K \subset \mathcal{D}_\epsilon$ be a compact set containing $z$. Let $\Gamma_K$ be the positively oriented contour around $[E_0, \infty)$ defined in (4.1). Introduce the following operators on $L^2(\mathbb{R}^3)$:

$$\mathcal{L}_{\infty,0}^{(\omega)}(\beta, b, z, \epsilon) := \frac{i}{2\pi} \int_{\Gamma_K} \mathrm{d}\xi\, \mathfrak{f}_\epsilon(\beta, z; \xi) R_\infty(b, \omega, \xi), \tag{1.14}$$

$$\mathcal{L}_{\infty,1}^{(\omega)}(\beta, b, z, \epsilon) := -\frac{i}{2\pi} \int_{\Gamma_K} \mathrm{d}\xi\ \mathfrak{f}_\epsilon(\beta, z; \xi) R_\infty(b, \omega, \xi) T_{1,\infty}(b, \omega, \xi), \tag{1.15}$$

$$\mathcal{L}_{\infty,2}^{(\omega)}(\beta, b, z, \epsilon) := \frac{i}{\pi} \int_{\Gamma_K} \mathrm{d}\xi\, \mathfrak{f}_\epsilon(\beta, z; \xi) R_\infty(b, \omega, \xi) \big((T_{1,\infty}(b, \omega, \xi))^2 - T_{2,\infty}(b, \omega, \xi)\big), \tag{1.16}$$

where $\mathfrak{f}_\epsilon(\beta, z; \cdot) := \ln(1 + \epsilon z \mathrm{e}^{-\beta \cdot})$. We will prove that generically these operators admit a jointly continuous integral kernel on $\mathbb{R}^6$ denoted by $\mathcal{L}_{\infty,n}^{(\omega)}(\cdot\,,\cdot\,;\beta,b,z,\epsilon)$, and are locally trace class.

Our results concerning the pressure and the magnetization are the following:

**Theorem 1.1.** *Suppose (E) and (R). Then:*
*i) $\mathbb{P}$-a.s. on $\Omega$, $\forall b \in \mathbb{R}$, $\forall 0 < \beta_1 < \beta_2$ and for any compact subset $K$ of $\mathcal{D}_\epsilon$, the thermodynamic limit of the pressure and magnetization exist. These limits are non-random, and:*

$$\mathcal{X}_{\infty,n}(\beta,b,z,\epsilon) := \lim_{L\to\infty} \mathcal{X}_{L,n}^{(\omega)}(\beta,b,z,\epsilon) = \left(\frac{q}{c}\right)^n \frac{\epsilon}{\beta}\mathbb{E}\big[\mathcal{L}_{\infty,n}^{(\omega)}(\mathbf{0},\mathbf{0};\beta,b,z,\epsilon)\big] \quad n = 0,1, \qquad (1.17)$$

*uniformly in $(\beta,z) \in [\beta_1,\beta_2]\times K$.*
*ii) $\mathbb{P}$-a.s. on $\Omega$, $\forall b \in \mathbb{R}$ and $\forall \beta > 0$, $P_\infty(\beta,b,\cdot\,,\epsilon) := \frac{\epsilon}{\beta}\mathbb{E}[\mathcal{L}_{\infty,0}^{(\omega)}(\mathbf{0},\mathbf{0};\beta,b,\cdot\,,\epsilon)]$ is an analytic function on $\mathcal{D}_\epsilon$. Moreover, $\forall 0 < \beta_1 < \beta_2$ and for any compact subset $K$ of $\mathcal{D}_\epsilon$, one has:*

$$\rho_\infty(\beta,b,z,\epsilon) := \lim_{L\to\infty}\rho_L^{(\omega)}(\beta,b,z,\epsilon) = \beta z \frac{\partial P_\infty}{\partial z}(\beta,b,z,\epsilon), \qquad (1.18)$$

*uniformly in $(\beta,z) \in [\beta_1,\beta_2]\times K$.*
*iii) $\mathbb{P}$-a.s. on $\Omega$, $\forall \beta > 0$, $\forall z \in \mathcal{D}_\epsilon$, $P_\infty(\beta,\cdot\,,z,\epsilon) \in \mathcal{C}^1(\mathbb{R})$, and $\mathcal{X}_{\infty,1}(\beta,b,z,\epsilon) = (\frac{q}{c})\frac{\partial P_\infty}{\partial b}(\beta,b,z,\epsilon)$.*

For the zero-field orbital susceptibility, we have:

**Theorem 1.2.** *Suppose also (E) and (R). Then:*
*i) $\mathbb{P}$-a.s. on $\Omega$, $\forall 0 < \beta_1 < \beta_2$ and for any compact subset $K$ of $\mathcal{D}_\epsilon$, the thermodynamic limit of the zero-field orbital susceptibility exists. The limit is non-random and it is given by:*

$$\mathcal{X}_{\infty,2}(\beta,0,z,\epsilon) := \lim_{L\to\infty}\mathcal{X}_{L,2}^{(\omega)}(\beta,0,z,\epsilon) = \left(\frac{q}{c}\right)^2 \frac{\epsilon}{\beta}\mathbb{E}\big[\mathcal{L}_{\infty,2}^{(\omega)}(\mathbf{0},\mathbf{0};\beta,0,z,\epsilon)\big], \qquad (1.19)$$

*uniformly in $(\beta,z) \in [\beta_1,\beta_2]\times K$.*
*ii) Let $\alpha \in [0,\frac{1}{4})$. Then $\mathbb{P}$-a.s. on $\Omega$, $\forall \beta > 0$ and $\forall z \in \mathcal{D}_\epsilon$, $P_\infty(\beta,\cdot\,,z,\epsilon)$ is a $\mathcal{C}^2$-function near $b = 0$, and $\mathcal{X}_{\infty,2}(\beta,0,z,\epsilon) = (\frac{q}{c})^2\frac{\partial^2 P_\infty}{\partial b^2}(\beta,0,z,\epsilon)$.*

**Remark 1.3.** $i$) Theorems 1.1 and 1.2 define the pressure, the magnetization and the susceptibility of the system, although the corresponding physical quantities, strictly speaking, require the strict positivity of the fugacity $z$.

$ii$) By using simple arguments, $P_\infty(\beta,-b,z,\epsilon) = \overline{P_\infty(\beta,b,z,\epsilon)} = P_\infty(\beta,b,z,\epsilon)$. Then Theorem 1.1 $iii$) implies that $\mathbb{P}$-a.s., $\forall\,\beta > 0$, $\forall\, z \in \mathcal{D}_\epsilon$, $\mathcal{X}_{\infty,1}(\beta,0,z,\epsilon) = 0$.

1.3. **Discussions and examples.** Let us comment (R1). Our approach is based on the representation of the finite-volume pressure by using the Dunford-Schwartz integral formula, see Section 4. However this requires the use of bounded below Schrödinger operators, see [12, Sect. VII.9]. As we allow realizations of $V_1^{(\omega)}$ to be negative with local singularities, this condition is fulfilled if $\mathbb{P}$-a.s. $V_1^{(\omega)} \in L^p_{\mathrm{uloc}}(\mathbb{R}^3)$ with $p > \frac{3}{2}$, see [23, Eq. (A21)]. The additional condition $p > 3$ will appear when estimating the derivative of the infinite-volume resolvent's integral kernel in (1.12), see the proof of Lemma 2.4 $ii$). Notice that (R1) does not cover Coulomb-type singularities and implies that $V_1^{(\omega)}$ roughly behaves locally like $\mathcal{O}(|\mathbf{x}|^{-(1-\epsilon)})$,

$\epsilon > 0$. Finally we mention that under the conditions on $V_1^{(\omega)}$ given in Section 1.2, there exists a $\omega$-independent constant $C > 0$ s.t. $\mathbb{P}$-a.s. on $\Omega$, [17]:

$$\|V_1^{(\omega)}\|_{p,\mathrm{uloc}} \leq C. \tag{1.20}$$

Let us discuss (1.3). Unlike $V_1^{(\omega)}$, we allow $V_2^{(\omega)}$ to have a polynomial growth at infinity. Notice that $V_2^{(\omega)}$ is not supposed to be monotone; indeed in that case the operator $H_\infty(b,\omega)$ may have only discrete spectrum, and then the problem becomes irrelevant. On the contrary, $V_2^{(\omega)}$ is basically of sparse barrier potential type, so that the corresponding operator has $[E_0,\infty)$ in its spectrum and with a non-trivial spectral type. By considering an arbitrary polynomial growth: $|\mathbf{x}|^\alpha$, $\alpha > 0$, the condition $(n+1)\alpha < 1$ with $n = 1,2$ will appear when proving the existence of the thermodynamic limits (1.17)-(1.19), see Proposition 6.2. However the relations given in Theorems 1.1 *iii*) and 1.2 *ii*) require the additional condition $(n+2)\alpha < 1$. Notice that if we only are interested in the case of the pressure or density, the assumption (R) can be relaxed to the ones ensuring the validity of the Pastur-Shubin formula for the integrated density of states, see [24, Sect. 2] and the method in [6, Sect. 3.4].

Let us discuss the assumption (E). Since the proof of Theorems 1.1 *i*) and 1.2 *i*) is based on the Birkhoff-Khintchine theorem in [21, Prop. 1.13], the use of $\mathbb{R}^3$-ergodic random field is crucial. Nevertheless we can replace the assumption (E) with:

(E') $V^{(\omega)}$ is a $\mathbb{Z}^3$-ergodic random field,

since this reduces to a $\mathbb{R}^3$-ergodic random field with the suspension technique, see [21].

We now give physically relevant examples covered by the assumptions (E)-(R) or (E')-(R).

**0**. *The non-negative Poisson random field.*

In (1.2) set $V_1^{(\omega)} = 0$ and choose for $V_2^{(\omega)}$ the random field with realizations given by:

$$V_P^{(\omega)}(\mathbf{x}) = \int_{\mathbb{R}^3} \boldsymbol{\mu}_\lambda^{(\omega)}(\mathrm{d}\mathbf{y})\, u(\mathbf{y}-\mathbf{x}) \quad \mathbf{x} \in \mathbb{R}^3,\ \omega \in \Omega,$$

where $\boldsymbol{\mu}_\lambda^{(\omega)}$ denotes the random Poisson measure on $\mathbb{R}^3$ with parameter $\lambda > 0$ and $u(\cdot) : \mathbb{R}^3 \to [0,\infty)$ is the single-site potential, see e.g. [21, 18]. By assuming that $u$ is compactly supported and $u \in L^\infty(\mathbb{R}^3)$, then (E) is satisfied as well as (R2), since we have $\mathbb{P}$-a.s., $\forall \mathbf{x} \in \mathbb{R}^3$, $0 \leq V_P^{(\omega)}(\mathbf{x}) \leq c(\omega)\ln(1+|\mathbf{x}|)$, see [13, Lem. 2.2].

**1**. *The alloy-type random field (the so-called 'Anderson potential').*

In (1.2) set $V_2^{(\omega)} = 0$ and choose for $V_1^{(\omega)}$ the random field with realizations given by:

$$V_A^{(\omega)}(\mathbf{x}) = g \sum_{\mathbf{j}\in\mathbb{Z}^3} \lambda_{\mathbf{j}}(\omega) u(\mathbf{x}-\mathbf{x}_{\mathbf{j}}) \quad \mathbf{x} \in \mathbb{R}^3,\ \omega \in \Omega, \quad g \in \mathbb{R}.$$

Here $\{\lambda_{\mathbf{j}}\}_{\mathbf{j}\in\mathbb{Z}^3}$ is a family of i.i.d. random variables with a common distribution what ensures (E'), see e.g. [21, 18]. Besides we suppose that $\forall \mathbf{j} \in \mathbb{Z}^3$, $|\lambda_{\mathbf{j}}(\omega)| \leq 1$ and $u$ satisfies the Birman-Solomyak condition: $\sum_{\mathbf{j}\in\mathbb{Z}^3}(\int_{\Lambda_1(\mathbf{j})} \mathrm{d}\mathbf{x}\, |u(\mathbf{x})|^p)^{\frac{1}{p}} < \infty$, $p > 3$, where $\Lambda_1(\mathbf{j})$ denotes the unit cube centered on site $\mathbf{j}$. Then $V_A^{(\omega)} \in L^p_{\mathrm{uloc}}(\mathbb{R}^3)$ with $p > 3$, and we have (see also [22]):

$$\mathcal{X}_{\infty,n}(\beta,b,z,\epsilon) = \left(\frac{q}{c}\right)^n \frac{\epsilon}{\beta|\Lambda_1(\mathbf{0})|} \int_{\Lambda_1(\mathbf{0})} \mathrm{d}\mathbf{x}\, \mathbb{E}\big[\mathcal{L}_{\infty,n}^{(\omega)}(\mathbf{x},\mathbf{x};\beta,b,z,\epsilon)\big] \quad n = 0,1,2. \tag{1.21}$$

**2**. *Further models.*

*The periodic case.* Set $V_2 = 0$. We assume that $V_1 \in L^p_{\mathrm{uloc}}(\mathbb{R}^3), p > 3$ and $\mathbb{Z}^3$-periodic. The suspension method can be applied to this case leading to define from $V_1$, a $\mathbb{R}^3$-ergodic random field $V_1^{(\omega)}$. So (1.21) holds true but in that case, $\mathbb{E}\big[\mathcal{L}^{(\omega)}_{\infty,n}(\cdot\,,\cdot\,;\beta,b,z,\epsilon)\big] = \mathcal{L}^{(\omega)}_{\infty,n}(\cdot\,,\cdot\,;\beta,z,\epsilon)$, see also [6, 22]. The *random displacements model* on $\mathbb{R}^3$ (see e.g. [18]) or *the quasi-periodic case* (see e.g. [21, 17]) are also covered by our results.

1.4. **The content.** In Section 2 we investigate the $\mathbb{P}$-a.s. analyticity (in the Hilbert-Schmidt topology) of the finite-volume resolvent and the analyticity of its integral kernel w.r.t. $b$.

In Section 3 we apply the gauge invariant magnetic perturbation theory (see e.g. [11, 20]) to the finite-volume perturbed resolvent. This allows us to get an expression of the partial derivatives w.r.t. $b$ of its integral kernel by keeping a good control over the linear growth of the magnetic vector potential. Although taking the resolvent as 'central object' leads to further technical difficulties, but it allows us to get more powerful results.

In Section 4 we formulate the finite-volume quantities by using the expressions of the derivatives w.r.t. $b$ of the kernel of the finite-volume resolvent obtained in the previous section.

In Section 5 we investigate the main properties on the infinite-volume operators involved in the thermodynamic limits. Note that here the gauge invariant perturbation theory gives $\mathbb{R}^3$-stationary quantities, see (1.1). This is necessary to apply the limit ergodic theorem.

In Section 6 we prove the almost-sure non-random thermodynamic limits by the Birkhoff-Khintchine theorem and we investigate the bulk properties. This section also contains the proof of Theorems 1.1 and 1.2.

## 2. Regularity of the finite-volume resolvent in the $b$-parameter

2.1. **Analyticity in the Hilbert-Schmidt topology.** Hereafter we will denote respectively by $\|\cdot\|_{\mathfrak{I}_1}$, $\|\cdot\|_{\mathfrak{I}_2}$ and $\|\cdot\|$, the trace norm in $\mathfrak{I}_1(L^2(\Lambda_L))$, the Hilbert-Schmidt (H-S) norm in $\mathfrak{I}_2(L^2(\Lambda_L))$, and the operator norm in $\mathfrak{B}(L^2(\Lambda_L))$.

From [6, Prop. 3.1], $\mathbb{P}$-a.s., $\forall b_0 \in \mathbb{R}$, $\forall 0 < L < \infty$ and $\forall \xi \in \mathbb{C}\backslash[E_0,\infty)$, there exists a complex neighborhood $\mathcal{V}_{\xi,L}(b_0)$ of $b_0$ s.t. the operator-valued function $\mathcal{V}_{\xi,L}(b_0) \ni b \mapsto R_L(b,\omega,\xi) := (H_L(b,\omega)-\xi)^{-1}$ is analytic in the H-S topology. We now precise this result.

Consider the following operators on $L^2(\Lambda_L)$:

$$S_{1,L}(b_0,\omega,\xi) := \mathbf{a}\cdot(i\nabla + b_0\mathbf{a})R_L(b_0,\omega,\xi), \quad S_{2,L}(b_0,\omega,\xi) := \frac{1}{2}\mathbf{a}^2 R_L(b_0,\omega,\xi). \tag{2.1}$$

Again from [6, Sect. 2], $\mathbb{P}$-a.s., $\forall b_0 \in \mathbb{R}$, $\forall 0 < L < \infty$ and $\forall \xi \in \mathbb{C}\setminus[E_0,\infty)$, they are bounded and, denote by $d(\xi) := \mathrm{dist}(\xi,[E_0,\infty))$, then there exists a constant $c > 0$ s.t.:

$$\|S_{1,L}(b_0,\omega,\xi)\| \leq c\left(1+\frac{1}{d(\xi)}\right)(1+|\xi|)L \quad \text{and} \quad \|S_{2,L}(b_0,\omega,\xi)\| \leq \frac{c}{d(\xi)}L^2. \tag{2.2}$$

For all integer $k \geq 1$, introduce the following family of H-S operators on $L^2(\Lambda_L)$:

$$J_{k,L}(\mathbf{i})(b_0,\omega,\xi) := R_L(b_0,\omega,\xi)\prod_{m=1}^{k} S_{i_m,L}(b_0,\omega,\xi), \quad \mathbf{i} = \{i_1,\ldots,i_k\} \in \{1,2\}^k, \tag{2.3}$$

and for $n \geq k \geq 1$, $\chi_k^n$, the characteristic function :

$$\chi_k^n(\mathbf{i}) := \begin{cases} 1 & \text{if } i_1+\cdots+i_k = n \\ 0 & \text{otherwise.}\end{cases}$$

**Proposition 2.1.** *$\mathbb{P}$-a.s. on $\Omega$, $\forall b_0 \in \mathbb{R}$, $\forall 0 < L < \infty$ and $\forall \xi \in \mathbb{C} \setminus [E_0, \infty)$, then there exists a complex neighborhood $\mathcal{V}_{\xi,L}(b_0)$ of $b_0$ s.t. in the H-S operators sense:*

$$\forall b \in \mathcal{V}_{\xi,L}(b_0), \quad R_L(b,\omega,\xi) = R_L(b_0,\omega,\xi) + \sum_{n=1}^{\infty} \frac{(b-b_0)^n}{n!} \frac{\partial^n R_L}{\partial b^n}(b_0,\omega,\xi), \tag{2.4}$$

*where for $n \geq 1$:*

$$\frac{\partial^n R_L}{\partial b^n}(b_0,\omega,\xi) := n! \sum_{k=1}^{n} (-1)^k \sum_{\mathbf{i} \in \{1,2\}^k} \chi_k^n(\mathbf{i}) J_{k,L}(\mathbf{i})(b_0,\omega,\xi). \tag{2.5}$$

*Proof.* Let $\xi \in \mathbb{C} \setminus [E_0,\infty)$, $0 < L < \infty$ and $(b, b_0) \in \mathbb{C} \times \mathbb{R}$. Set:

$$S_L(b,b_0,\omega,\xi) := \delta b\, S_{1,L}(b_0,\omega,\xi) + (\delta b)^2 S_{2,L}(b_0,\omega,\xi), \quad \delta b := b - b_0. \tag{2.6}$$

From the $n$-th iterated second resolvent equation, one has:

$$R_L(b,\omega,\xi) = R_L(b_0,\omega,\xi)\left[\mathbb{I} + \sum_{k=1}^{n} (-1)^k \big(S_L(b,b_0,\omega,\xi)\big)^k\right] + (-1)^{n+1} R_L(b,\omega,\xi)\big(S_L(b,b_0,\omega,\xi)\big)^{n+1}.$$

Then by (2.6), one gets after some rearranging:

$$R_L(b,\omega,\xi) = R_L(b_0,\omega,\xi) + \sum_{k=1}^{n} (\delta b)^k \sum_{l=1}^{k} (-1)^l \sum_{\mathbf{i} \in \{1,2\}^l} \chi_l^k(\mathbf{i}) J_{l,L}(\mathbf{i})(b_0,\omega,\xi) + \mathcal{S}_{n+1,L}(b,b_0,\omega,\xi), \tag{2.7}$$

$$\text{with:} \quad \mathcal{S}_{n+1,L}(b,b_0,\omega,\xi) := (\delta b)^{n+1} \Bigg\{ \sum_{k=0}^{n-1} (\delta b)^k \sum_{l=1}^{n} (-1)^l \sum_{\mathbf{i} \in \{1,2\}^l} \chi_l^{k+n+1}(\mathbf{i}) J_{l,L}(\mathbf{i})(b_0,\omega,\xi) +$$

$$+ (-1)^{n+1} \sum_{k=0}^{n+1} (\delta b)^k \sum_{\mathbf{i} \in \{1,2\}^{n+1}} \chi_{n+1}^{k+n+1}(\mathbf{i}) R_L(b,\omega,\xi) \prod_{m=1}^{n+1} S_{i_m,L}(b_0,\omega,\xi) \Bigg\}. \tag{2.8}$$

By using the analyticity properties of the resolvent in the H-S topology given above, the proposition follows from (2.7) and (2.8). ∎

2.2. **Regularity of the integral kernel.** We know from [23, Thm. B.7.2] that $\mathbb{P}$-a.s., $\forall b \in \mathbb{R}$, $\forall L \in (0,\infty]$, $\forall \eta > 0$ and $\forall \xi \in \mathbb{C}$, $d(\xi) \geq \eta$, $R_L(b,\omega,\xi)$ has an integral kernel $R_L^{(1)}(\cdot\,,\cdot\,; b,\omega,\xi)$ jointly continuous on $\Lambda_L^2 \setminus D_L$, $D_L := \{(\mathbf{x},\mathbf{y}) \in \Lambda_L^2 : \mathbf{x} = \mathbf{y}\}$. Moreover $\mathbb{P}$-a.s., $\forall \eta > 0$ there exists a constant $\gamma = \gamma(\eta) > 0$ and a polynomial $p(\cdot)$ s.t. $\forall b \in \mathbb{R}$, $\forall L \in (0,\infty]$ and $\forall \xi \in \mathbb{C}$, $d(\xi) \geq \eta$:

$$\forall\, (\mathbf{x},\mathbf{y}) \in \Lambda_L^2 \setminus D_L, \quad |R_L^{(1)}(\mathbf{x},\mathbf{y}; b,\omega,\xi)| \leq |p(\xi)| \frac{\mathrm{e}^{-\gamma_\xi |\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|} \quad \text{with } \gamma_\xi := \frac{\gamma}{1+|\xi|}. \tag{2.9}$$

Notice that $\gamma_\xi$ can be more explicit w.r.t. the energy parameter. The one given in (2.9), valid for $\xi \in \mathbb{C}, d(\xi) \geq \eta > 0$, contains the $\xi$-dependence we need in the whole of this work.

**Remark 2.2.** Consequently the product $\Pi_{l=1}^m R_L(b,\omega,\xi_l)$, $m \geq 2$ has an integral kernel jointly continuous on $\Lambda_L^2$, and moreover, $\mathbb{P}$-a.s., $\forall \eta > 0$, there exists a constant $\gamma = \gamma(\eta, m) > 0$ and a polynomial $p(\cdot)$ s.t. $\forall b \in \mathbb{R}$, $\forall L \in (0,\infty]$ and $\forall \xi_l \in \mathbb{C}, l = 1,\dots,m$, $d(\xi_l) \geq \eta$:

$$\forall\, (\mathbf{x},\mathbf{y}) \in \Lambda_L^2, \quad \left|\Big(\Pi_{l=1}^m R_L(b,\omega,\xi_l)\Big)(\mathbf{x},\mathbf{y})\right| \leq (\Pi_{l=1}^m |p(\xi_l)|) \mathrm{e}^{-\gamma_\xi |\mathbf{x}-\mathbf{y}|}, \quad \gamma_\xi := \frac{\gamma}{1+|\xi|}. \tag{2.10}$$

This follows by induction on $m$ from the case of $m = 2$ with:

$$\left(\Pi_{l=1}^{m} R_L(b,\omega,\xi_l)\right)(\mathbf{x},\mathbf{y}) = \int_{\Lambda_L} \mathrm{d}\mathbf{z}\, \left(\Pi_{l=1}^{m-1} R_L(b,\omega,\xi_l)\right)(\mathbf{x},\mathbf{z}) R_L^{(1)}(\mathbf{z},\mathbf{y};b,\omega,\xi_m).$$

When $m = 2$ the continuity property holds true since the kernel $R_L^{(1)}(\cdot\,,\cdot\,;b,\omega,\xi)$ fulfills the assumptions of Lemma 7.1. Furthermore from (2.9) and (7.12), we get (2.10).

From Proposition 2.1 together with these results, we now prove:

**Proposition 2.3.** *$\mathbb{P}$-a.s. on $\Omega$, $\forall b_0 \in \mathbb{R}$, $\forall 0 < L < \infty$, $\forall \eta > 0$ and $\forall \xi \in \mathbb{C}$, $d(\xi) \geq \eta$, then there exists a complex neighborhood $\nu_{\xi,L,\omega}(b_0)$ of $b_0$ s.t. $\forall (\mathbf{x},\mathbf{y}) \in \Lambda_L^2 \setminus D_L$, $b \mapsto R_L^{(1)}(\mathbf{x},\mathbf{y};b,\omega,\xi)$ is an analytic function on $\nu_{\xi,L,\omega}(b_0)$ .*

An important point for the proof of Proposition 2.3 is the following estimate. We choose to give its proof in the appendix of the paper, see Section 7.

**Lemma 2.4.** *i) $\mathbb{P}$-a.s. on $\Omega$, $\forall b \in \mathbb{R}$, $\forall L \in (0,\infty]$, $\forall \eta > 0$ and $\forall \xi \in \mathbb{C}$, $d(\xi) \geq \eta$, then $(i\nabla_{\mathbf{x}} + b\mathbf{a}(\mathbf{x}))R_L(b,\omega,\xi)$ has an integral kernel jointly continuous on $\Lambda_L^2 \setminus D_L$.*
*ii) $\mathbb{P}$-a.s. on $\Omega$, $\forall \eta > 0$, there exists a constant $\gamma = \gamma(\eta) > 0$ and a polynomial $p(\cdot\,)$ s.t. $\forall L \in (0,\infty]$, $\forall b \in \mathbb{R}$, $\forall \xi \in \mathbb{C}$, $d(\xi) \geq \eta$, one has on $\Lambda_L^2 \setminus D_L$:*

$$|(i\nabla_{\mathbf{x}} + b\mathbf{a}(\mathbf{x}))R_L^{(1)}(\mathbf{x},\mathbf{y};b,\omega,\xi)| \leq (1+|b|)^3 |p(\xi)|(1+|\mathbf{x}|^\alpha + |\mathbf{y}|^\alpha)\frac{\mathrm{e}^{-\gamma_\xi|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|^2}, \quad \gamma_\xi := \frac{\gamma}{1+|\xi|}. \tag{2.11}$$

*Proof of Proposition 2.3.* Under the conditions of the proposition, from (2.9) and Lemma 2.4, the operators in (2.1) have an integral kernel jointly continuous on $\Lambda_L^2 \setminus D_L$, given by:

$$S_{1,L}(\mathbf{x},\mathbf{y};b_0,\omega,\xi) := \mathbf{a}(\mathbf{x}) \cdot (i\nabla_{\mathbf{x}} + b_0\mathbf{a}(\mathbf{x}))R_L^{(1)}(\mathbf{x},\mathbf{y};b_0,\omega,\xi),$$
$$S_{2,L}(\mathbf{x},\mathbf{y};b_0,\omega,\xi) := \frac{1}{2}\mathbf{a}^2(\mathbf{x})R_L^{(1)}(\mathbf{x},\mathbf{y};b_0,\omega,\xi),$$

and moreover there exists a constant $\gamma = \gamma(\eta) > 0$ and a polynomial $p(\cdot\,)$ s.t. on $\Lambda_L^2 \setminus D_L$:

$$|S_{j,L}(\mathbf{x},\mathbf{y};b_0,\omega,\xi)| \leq |p(\xi)|(1+L^\alpha)L^j \frac{\mathrm{e}^{-\gamma_\xi|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|^2}, \quad j = 1,2. \tag{2.12}$$

Then, considering (2.5), $\mathbb{P}$-a.s., $\forall b_0 \in \mathbb{R}$, $\forall 0 < L < \infty$, $\forall \eta > 0$ and $\forall \xi \in \mathbb{C}$, $d(\xi) \geq \eta$, the operator $\frac{\partial^n R_L}{\partial b^n}(b_0,\omega,\xi)$, $n \geq 1$ has an integral kernel given on $\Lambda_L^2 \setminus D_L$ by:

$$\frac{\partial^n R_L}{\partial b^n}(\mathbf{x},\mathbf{y};b_0,\omega,\xi) := n! \sum_{k=1}^{n} (-1)^k \sum_{\mathbf{i} \in \{1,2\}^k} \chi_k^n(\mathbf{i}) J_{k,L}(\mathbf{i})(\mathbf{x},\mathbf{y};b_0,\omega,\xi), \tag{2.13}$$

where $J_{k,L}(\mathbf{i})(\cdot\,,\cdot\,;b_0,\omega,\xi)$ stands for the integral kernel of the operator in (2.3). It reads as:

$$J_{k,L}(\mathbf{i})(\mathbf{x},\mathbf{y};b_0,\omega,\xi) := \int_{\Lambda_L} \mathrm{d}\mathbf{z}_1 \cdots \int_{\Lambda_L} \mathrm{d}\mathbf{z}_k\, R_L^{(1)}(\mathbf{x},\mathbf{z}_1;b_0,\omega,\xi) \times$$
$$\times S_{i_1,L}(\mathbf{z}_1,\mathbf{z}_2;b_0,\omega,\xi) \cdots S_{i_k,L}(\mathbf{z}_k,\mathbf{y};b_0,\omega,\xi).$$

Furthermore, from estimates (2.9) and (2.12), by applying $k$-times successively Lemma 7.1 *i)* combined with (7.14), we obtain that $J_{k,L}(\mathbf{i})(\cdot\,,\cdot\,;b_0,\omega,\xi)$ is jointly continuous on $\Lambda_L^2 \setminus D_L$.

By using (2.9), (2.12) with Lemma 7.2 *ii*), $\mathbb{P}$-a.s., $\forall b_0 \in \mathbb{R}$, $\forall \eta > 0$, there exists a constant $\gamma = \gamma(\eta) > 0$ and a polynomial $p(\cdot)$ s.t. $\forall 0 < L < \infty$, $\forall \xi \in \mathbb{C}$, $d(\xi) \geq \eta$ and $\forall (\mathbf{x}, \mathbf{y}) \in \Lambda_L^2 \setminus D_L$:

$$|J_{k,L}(\mathbf{i})(\mathbf{x}, \mathbf{y}; b_0, \omega, \xi)| \leq |p(\xi)|^k (1 + L^\alpha)^k L^{i_1 + \cdots + i_k} \frac{\mathrm{e}^{-\frac{\gamma_\xi}{2^k}|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x} - \mathbf{y}|}, \quad \gamma_\xi := \frac{\gamma}{1 + |\xi|}.$$

This estimate then imply the following rough estimate which holds on $\Lambda_L^2 \setminus D_L$:

$$\forall n \in \mathbb{N}^*, \quad \frac{1}{n!}\left|\frac{\partial^n R_L}{\partial b^n}(\mathbf{x}, \mathbf{y}; b_0, \omega, \xi)\right| \leq c^n |p(\xi)|^n (1 + L^\alpha)^n L^n \frac{\mathrm{e}^{-\frac{\gamma_\xi}{2^n}|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x} - \mathbf{y}|}, \tag{2.14}$$

for some constant $c > 0$. So the analyticity property follows from (2.4), (2.14) since for $|b - b_0|$ sufficiently small, the corresponding Taylor expansion converges. Here we use,

$$\sup_{\mathbf{x} \in \Lambda_L} \int_{\mathbb{R}^3} \mathrm{d}\mathbf{y} \, \frac{\mathrm{e}^{-\frac{\varsigma}{2^n}|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x} - \mathbf{y}|} = \left(\frac{2^n}{\varsigma}\right)^2, \quad \varsigma > 0. \tag{2.15}$$

■

## 3. A new expression for the first and second derivatives w.r.t. $b$

In order to determine the thermodynamic limits, we want to isolate in the expression (2.13) the term giving rise to the growth w.r.t. $L$ when $L \to \infty$, see (2.14).

Let $\mathbf{x}, \mathbf{y} \in \Lambda_L$. Define the magnetic phase $\phi$ as:

$$\phi(\mathbf{x}, \mathbf{y}) := \frac{1}{2}\mathbf{e}_3 \cdot (\mathbf{y} \wedge \mathbf{x}) = -\phi(\mathbf{y}, \mathbf{x}) \quad \text{with } \mathbf{e}_3 := (0, 0, 1). \tag{3.1}$$

Introduce on $L^2(\Lambda_L)$ the operators $T_{j,L}(b, \omega, \xi)$, $j = 1, 2$ defined via their integral kernel:

$$\forall (\mathbf{x}, \mathbf{y}) \in \Lambda_L^2 \setminus D_L, \quad T_{1,L}(\mathbf{x}, \mathbf{y}; b, \omega, \xi) := \mathbf{a}(\mathbf{x} - \mathbf{y}) \cdot (i\nabla_{\mathbf{x}} + b\mathbf{a}(\mathbf{x}))R_L^{(1)}(\mathbf{x}, \mathbf{y}; b, \omega, \xi), \tag{3.2}$$

$$T_{2,L}(\mathbf{x}, \mathbf{y}; b, \omega, \xi) := \frac{1}{2}\mathbf{a}^2(\mathbf{x} - \mathbf{y})R_L^{(1)}(\mathbf{x}, \mathbf{y}; b, \omega, \xi). \tag{3.3}$$

Obviously $|\mathbf{a}(\mathbf{x} - \mathbf{y})| \leq |\mathbf{x} - \mathbf{y}|$, then from (2.9) and (2.11), $\mathbb{P}$-a.s., $\forall b \in \mathbb{R}$, $\forall \eta > 0$, there exists $\gamma = \gamma(\eta) > 0$ and a polynomial $p(\cdot)$ s.t. $\forall 0 < L < \infty$ and $\forall \xi \in \mathbb{C}$, $d(\xi) \geq \eta$:

$$|T_{j,L}(\mathbf{x}, \mathbf{y}; b, \omega, \xi)| \leq |p(\xi)|(1 + L^\alpha)\frac{\mathrm{e}^{-\gamma_\xi|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x} - \mathbf{y}|} \quad j = 1, 2. \tag{3.4}$$

Hence $T_{j,L}(b, \omega, \xi)$, $j = 1, 2$ are bounded operators and

$$\|T_{j,L}(b, \omega, \xi)\| \leq |p(\xi)|(1 + L^\alpha) \quad j = 1, 2, \tag{3.5}$$

for some polynomial $p(\cdot)$. For any $k \in \{1, 2\}$ and $m \in \{0, 1\}$, define on $\Lambda_L^2$:

$$\begin{aligned}\mathcal{T}_{k,L}^m(\mathbf{x}, \mathbf{y}; b, \omega, \xi) := \sum_{j=1}^k (-1)^j \sum_{\mathbf{i} \in \{1,2\}^j} \chi_j^k(\mathbf{i}) \int_{\Lambda_L} \mathrm{d}\mathbf{z}_1 \cdots \int_{\Lambda_L} \mathrm{d}\mathbf{z}_j \left(i\phi(\mathbf{z}_j, \mathbf{y}) - i\phi(\mathbf{z}_j, \mathbf{x})\right)^m \times \\ \times R_L^{(1)}(\mathbf{x}, \mathbf{z}_1; b, \omega, \xi) T_{i_1,L}(\mathbf{z}_1, \mathbf{z}_2; b, \omega, \xi) \cdots T_{i_j,L}(\mathbf{z}_j, \mathbf{y}; b, \omega, \xi).\end{aligned} \tag{3.6}$$

Here we set $0^0 = 1$. Notice that for $\mathbf{x} = \mathbf{y}$, the terms in the r.h.s. of (3.6) containing the magnetic phase vanish. Clearly $\mathbb{P}$-a.s., $\forall b \in \mathbb{R}$, $\forall 0 < L < \infty$, $\forall \eta > 0$ and $\forall \xi \in \mathbb{C}$,

$d(\xi) \geq \eta$, $\mathcal{T}^m_{k,L}(\cdot\,,\cdot\,; b,\omega,\xi)$ is jointly continuous on $\Lambda_L^2$. To see that, we apply $j$-times Lemma 7.1 considering (2.9), (3.4) and (7.13). This also gives:

$$\forall (\mathbf{x},\mathbf{y}) \in \Lambda_L^2, \quad |\mathcal{T}^m_{k,L}(\mathbf{x},\mathbf{y}; b,\omega,\xi)| \leq |p(\xi)|^k L^m (1+L^\alpha)^k, \quad k \in \{1,2\},\, m \in \{0,1\}, \tag{3.7}$$

for some polynomial $p(\cdot\,)$. Note also that when $\mathbf{x} = \mathbf{y}$, the r.h.s. of (3.7) behaves like $L^{\alpha k}$.

We now formulate the main result of this section; its proof is given in the next subsections:

**Proposition 3.1.** *$\mathbb{P}$-a.s. on $\Omega$, $\forall b \in \mathbb{R}$, $\forall 0 < L < \infty$, $\forall \eta > 0$ and $\forall \xi \in \mathbb{C}$, $d(\xi) \geq \eta$, then $\forall (\mathbf{x},\mathbf{y}) \in \Lambda_L^2 \setminus D_L$ and for $n = 1,2$:*

$$\frac{1}{n!}\frac{\partial^n R_L^{(1)}}{\partial b^n}(\mathbf{x},\mathbf{y}; b,\omega,\xi) = \frac{\big(i\phi(\mathbf{x},\mathbf{y})\big)^n}{n!} R_L^{(1)}(\mathbf{x},\mathbf{y}; b,\omega,\xi) + \sum_{k=1}^n \mathcal{T}^{n-k}_{k,L}(\mathbf{x},\mathbf{y}; b,\omega,\xi). \tag{3.8}$$

3.1. **Some preliminary results.** Let $(b,b_0) \in \mathbb{R}^2$ and set $\delta b := b - b_0$. Let $\eta > 0$ and $\xi \in \mathbb{C}$, $d(\xi) \geq \eta$. Introduce on $L^2(\Lambda_L)$ the operators $\tilde{R}_L(b,b_0,\omega,\xi)$ and $\tilde{T}_{j,L}(b,b_0,\omega,\xi)$, $j = 1,2$ through their integral kernel which are respectively defined by:

$$\forall\,(\mathbf{x},\mathbf{y}) \in \Lambda_L^2 \setminus D_L, \quad \tilde{R}_L^{(1)}(\mathbf{x},\mathbf{y}; b,b_0,\omega,\xi) := \mathrm{e}^{i\delta b\phi(\mathbf{x},\mathbf{y})} R_L^{(1)}(\mathbf{x},\mathbf{y}; b_0,\omega,\xi), \tag{3.9}$$

$$\tilde{T}_{j,L}(\mathbf{x},\mathbf{y}; b,b_0,\omega,\xi) := \mathrm{e}^{i\delta b\phi(\mathbf{x},\mathbf{y})} T_{j,L}(\mathbf{x},\mathbf{y}; b_0,\omega,\xi). \tag{3.10}$$

Set also:

$$\tilde{T}_L(b,b_0,\omega,\xi) := \delta b\,\tilde{T}_{1,L}(b,b_0,\omega,\xi) + (\delta b)^2 \tilde{T}_{2,L}(b,b_0,\omega,\xi). \tag{3.11}$$

Except a gauge phase factor, the integral kernel of $\tilde{R}_L(b,b_0,\omega,\xi)$ and $\tilde{T}_{j,L}(b,b_0,\omega,\xi)$ is the same as the one of $R_L(b_0,\omega,\xi)$ and $T_{j,L}(b_0,\omega,\xi)$ respectively. Therefore, $\mathbb{P}$-a.s., $\forall (b_0,b) \in \mathbb{R}^2$, $\forall 0 < L < \infty$, $\forall \eta > 0$ and $\forall \xi \in \mathbb{C}$, $d(\xi) \geq \eta$, they are bounded operators with a norm satisfying (3.5). Besides they are eventually H-S operators on $L^2(\Lambda_L)$, and:

$$\|\tilde{R}_L(b,b_0,\omega,\xi)\|_{\mathfrak{I}_2} \leq |p(\xi)| L^{\frac{3}{2}}, \quad \|\tilde{T}_{j,L}(b,b_0,\omega,\xi)\|_{\mathfrak{I}_2} \leq |p(\xi)|(1+L^\alpha) L^{\frac{3}{2}}. \tag{3.12}$$

Under the same conditions as above, introduce the following bounded operators on $L^2(\Lambda_L)$:

$$\tilde{\mathcal{T}}_{1,L}(b,b_0,\omega,\xi) := -\tilde{R}_L(b,b_0,\omega,\xi)\tilde{T}_{1,L}(b,b_0,\omega,\xi), \tag{3.13}$$

$$\tilde{\mathcal{T}}_{2,L}(b,b_0,\omega,\xi) := \tilde{R}_L(b,b_0,\omega,\xi)\Big(\big(\tilde{T}_{1,L}(b,b_0,\omega,\xi)\big)^2 - \tilde{T}_{2,L}(b,b_0,\omega,\xi)\Big), \tag{3.14}$$

$$\tilde{\mathcal{T}}_{3,L}(b,b_0,\omega,\xi) := (\delta b)^3 \sum_{k=0}^{1} (\delta b)^k \sum_{\mathbf{i} \in \{1,2\}^2} \chi_2^{3+k}(\mathbf{i}) \tilde{R}_L(b,b_0,\omega,\xi)\tilde{T}_{i_1,L}(b,b_0,\omega,\xi)\tilde{T}_{i_2,L}(b,b_0,\omega,\xi) + \\ - R_L(b,\omega,\xi)\big(\tilde{T}_L(b,b_0,\omega,\xi)\big)^3. \tag{3.15}$$

Then we prove (see also [4, Lem. 3.2] and [11, proof of Prop. 3.2]):

**Lemma 3.2.** *$\mathbb{P}$-a.s. on $\Omega$, $\forall (b,b_0) \in \mathbb{R}^2$, $\forall 0 < L < \infty$, $\forall \eta > 0$ and $\forall \xi \in \mathbb{C}$, $d(\xi) \geq \eta$, then one has in the H-S operators sense:*

$$R_L(b,\omega,\xi) = \tilde{R}_L(b,b_0,\omega,\xi) + \sum_{k=1}^{2} (\delta b)^k \tilde{\mathcal{T}}_{k,L}(b,b_0,\omega,\xi) + \tilde{\mathcal{T}}_{3,L}(b,b_0,\omega,\xi). \tag{3.16}$$

*Proof.* From [10, Sect. 2], $\mathbb{P}$-a.s, $\forall b \in \mathbb{R}$, $\tilde{\mathrm{D}} := \{\varphi \in \mathcal{C}^1(\overline{\Lambda_L}) \cap \mathcal{C}^2(\Lambda_L), \varphi|_{\partial\Lambda_L} = 0\}$ is a core for $H_L(b,\omega)$. Since $\mathbf{a}(\cdot)$ is the symmetric gauge, one has in the form sense on $\tilde{\mathrm{D}} \times \tilde{\mathrm{D}}$:

$$(-i\nabla_{\mathbf{x}} - b\mathbf{a}(\mathbf{x}))\mathrm{e}^{i\delta b\phi(\mathbf{x},\mathbf{y})} = \mathrm{e}^{i\delta b\phi(\mathbf{x},\mathbf{y})}(-i\nabla_{\mathbf{x}} - b_0\mathbf{a}(\mathbf{x}) - \delta b\mathbf{a}(\mathbf{x}-\mathbf{y})). \tag{3.17}$$

In view of (3.9)-(3.11) and (3.17), we get for any $(\varphi,\psi) \in \tilde{\mathrm{D}} \times \mathcal{C}_0^\infty(\Lambda_L)$:

$$l_L(\varphi,\psi) := \langle (H_L(b,\omega) - \overline{\xi})\varphi, \tilde{R}_L(b,b_0,\omega,\xi)\psi\rangle = \langle\varphi,\psi\rangle + \langle\varphi, \tilde{T}_L(b,b_0,\omega,\xi)\psi\rangle.$$

By standard arguments, $l_L$ can be extended in a bounded form on $D(H_L(b,\omega)) \times L^2(\Lambda_L)$. Let $\varphi = R_L(b,\omega,\bar{\xi})\hat{\varphi}$, with $\hat{\varphi} \in L^2(\Lambda_L)$. Then the following identity holds in $\mathfrak{B}(L^2(\Lambda_L))$, and eventually in the H-S operators sense (see (3.12)):

$$R_L(b,\omega,\xi) = \tilde{R}_L(b,b_0,\omega,\xi) - R_L(b,\omega,\xi)\tilde{T}_L(b,b_0,\omega,\xi). \tag{3.18}$$

Now we iterate twice (3.18), in view of (3.11) and (3.13)-(3.15) the lemma follows. ∎

3.2. **Proof of Proposition 3.1.** Following Lemma 3.2, and by rewriting (3.16) in terms of corresponding integral kernels, we get on $\Lambda_L^2 \setminus D_L$:

$$R_L^{(1)}(\mathbf{x},\mathbf{y};b,\omega,\xi) = \tilde{R}_L^{(1)}(\mathbf{x},\mathbf{y};b,b_0,\omega,\xi) + \sum_{k=1}^{2}(\delta b)^k\tilde{\mathcal{T}}_{k,L}(\mathbf{x},\mathbf{y};b,b_0,\omega,\xi) + \tilde{\mathcal{T}}_{3,L}(\mathbf{x},\mathbf{y};b,b_0,\omega,\xi), \tag{3.19}$$

where, for all integer $k \in \{1,2\}$ and for any $(\mathbf{x},\mathbf{y}) \in \Lambda_L^2$:

$$\begin{aligned}\tilde{\mathcal{T}}_{k,L}(\mathbf{x},\mathbf{y};b,b_0,\omega,\xi) := \sum_{j=1}^{k}(-1)^j \sum_{\mathbf{i}\in\{1,2\}^j}\chi_j^k(\mathbf{i})\int_{\Lambda_L^j}\mathrm{d}\mathbf{z}_1\cdots\mathrm{d}\mathbf{z}_j\,\mathrm{e}^{i\delta b(\phi(\mathbf{x},\mathbf{z}_1)+\cdots+\phi(\mathbf{z}_j,\mathbf{y}))}\times\\ \times R_L^{(1)}(\mathbf{x},\mathbf{z}_1;b_0,\omega,\xi)T_{i_1,L}(\mathbf{z}_1,\mathbf{z}_2;b_0,\omega,\xi)\cdots T_{i_j,L}(\mathbf{z}_j,\mathbf{y};b_0,\omega,\xi),\end{aligned} \tag{3.20}$$

and $\tilde{\mathcal{T}}_{3,L}(\cdot\,,\cdot\,;b,b_0,\omega,\xi)$ stands for the kernel of $\tilde{\mathcal{T}}_{3,L}(b,b_0,\omega,\xi)$, see (3.15). We now remove the $b$-dependence of the coefficient of $(\delta b)^k$ in the sum (3.19) by expanding in Taylor series the exponential phase factor in (3.9) and in (3.20) up to the second order. Thus on $\Lambda_L^2 \setminus D_L$:

$$\begin{aligned}\tilde{R}_L^{(1)}(\mathbf{x},\mathbf{y};b,b_0,\omega,\xi) + \sum_{k=1}^{2}(\delta b)^k\tilde{\mathcal{T}}_{k,L}(\mathbf{x},\mathbf{y};b,b_0,\omega,\xi) = \sum_{k=0}^{2}(\delta b)^k\frac{\big(i\phi(\mathbf{x},\mathbf{y})\big)^k}{k!}R_L^{(1)}(\mathbf{x},\mathbf{y};b_0,\omega,\xi) +\\ + \sum_{k=1}^{2}(\delta b)^k\sum_{m=1}^{k}\mathcal{T}_{m,L}^{k-m}(\mathbf{x},\mathbf{y};b_0,\omega,\xi) + \mathcal{T}_{3,L}(\mathbf{x},\mathbf{y};b,b_0,\omega,\xi),\end{aligned}$$

where by construction the remainder term $\mathcal{T}_{3,L}(\mathbf{x},\mathbf{y};\cdot\,,b_0,\omega,\xi)$ satisfies the property that its first and second derivatives at $b_0$ are all zero. It remains to use the definitions (3.9)-(3.11) combined with (2.9), (3.4) and Lemma 7.2, this shows that $\mathbb{P}$-a.s., $\forall(b,b_0) \in \mathbb{R}^2$, $\forall 0 < L < \infty$, $\forall\eta > 0$ and $\forall\xi \in \mathbb{C}$, $d(\xi) \geq \eta$, $|\tilde{\mathcal{T}}_{3,L}(\mathbf{x},\mathbf{y};b,b_0,\omega,\xi)| = \mathcal{O}(|\delta b|^3)$ when $|\delta b| \to 0$ uniformly in $\mathbf{x},\mathbf{y} \in \Lambda_L$. Then the proposition follows from Proposition 2.3. ∎

## 4. The finite-volume diamagnetic response

Here, by using the results of Section 3, we want to get a new expression for the magnetization and the susceptibility. Recall that by applying [6, Thm 1.1 *i*)], $\mathbb{P}$-a.s., $\forall\beta > 0$ and $\forall b \in \mathbb{R}$, the pressure defined in (1.9) has an analytic extension in $z \in \mathcal{D}_\epsilon$ (see (1.8)). This analytic continuation is defined as the following.

Let $\beta > 0$ and $K \subset \mathcal{D}_\epsilon$ be a compact subset. Let $\Gamma_K$ be the positively oriented contour around the interval $[E_0, \infty)$ defined by:

$$\Gamma_K := \{\Re\xi = E_K,\ \Im\xi \in [\frac{-\vartheta_K}{2\beta}, \frac{\vartheta_K}{2\beta}]\} \cup \{\Re\xi \in [E_K, \xi_K),\ \Im\xi = \pm\frac{\vartheta_K}{2\beta}\} \cup \\ \cup \{\Re\xi \geq \xi_K,\ \arg(\xi - \xi_K \mp i\frac{\vartheta_K}{2\beta}) = \pm\varsigma\}. \quad (4.1)$$

The constants $\vartheta_K > 0$, $E_K < E_0$, $\varsigma \in (0, \pi/2)$ and $\xi_K > E_0$ are chosen so that for all $z \in K$, the closed subset surrounding by $\Gamma_K$ is a strict subset of the holomorphic domain of the map $\xi \in \mathbb{C} \mapsto \mathfrak{f}_\epsilon(\beta, z; \xi) := \ln(1 + \epsilon z \mathrm{e}^{-\beta\xi})$, see [6, Lem. 3.4]. Besides $\mathfrak{f}_\epsilon(\beta, z; \cdot\,)$ admits an exponentially decreasing estimate on $\Gamma_K$, i.e. there exists a constant $c = c(\beta, K) > 0$ s.t.:

$$\forall z \in K,\ \forall \xi \in \Gamma_K, \quad |\mathfrak{f}_\epsilon(\beta, z; \xi)| \leq c\mathrm{e}^{-\beta\Re\xi}. \quad (4.2)$$

Let $\beta > 0$, $b \in \mathbb{R}$, $0 < L < \infty$, $z \in \mathcal{D}_\epsilon$, $K \subset \mathcal{D}_\epsilon$ be a compact neighborhood of $z$ and $\Gamma_K$ given in (4.1). Introduce on $L^2(\Lambda_L)$:

$$\mathcal{L}_L^{(\omega)}(\beta, b, z, \epsilon) := \frac{i}{2\pi} \int_{\Gamma_K} \mathrm{d}\xi\, \mathfrak{f}_\epsilon(\beta, z; \xi) R_L(b, \omega, \xi). \quad (4.3)$$

Then $\mathbb{P}$-a.s., $\forall \beta > 0$, $\forall b \in \mathbb{R}$, $\forall 0 < L < \infty$ and $\forall z \in \mathcal{D}_\epsilon \cap \mathbb{R}$, (4.3) defines a trace class operator on $L^2(\Lambda_L)$ (see [6, Sect. 3.2]), and via the standard functional calculus we have, $\mathcal{L}_L^{(\omega)}(\beta, b, z, \epsilon) = \ln(\mathbb{I} + \epsilon z \mathrm{e}^{-\beta H_L(b,\omega)})$. Hence, this allows us to define the finite-volume pressure as:

$$P_L^{(\omega)}(\beta, b, z, \epsilon) = \frac{\epsilon}{\beta|\Lambda_L|} \mathrm{Tr}_{L^2(\Lambda_L)}\Big(\mathcal{L}_L^{(\omega)}(\beta, b, z, \epsilon)\Big). \quad (4.4)$$

It is shown in [6, Sect. 3.3] that (4.4) can be analytically extended to any $z \in \mathcal{D}_\epsilon$, and on the other hand, the definition (4.4) is independent of the choice of the compact subset $K$.

**Proposition 4.1.** *$\mathbb{P}$-a.s. on $\Omega$, $\forall 0 < L < \infty$, $\forall \beta > 0$, $\forall b \in \mathbb{R}$, $\forall z \in \mathcal{D}_\epsilon$ and for any compact subset $K$ of $\mathcal{D}_\epsilon$ s.t. $z \in K$, then one has for $n = 1, 2$:*

$$\mathcal{X}_{L,n}^{(\omega)}(\beta, b, z, \epsilon) = \left(\frac{q}{c}\right)^n \frac{\epsilon}{\beta|\Lambda_L|} \mathrm{Tr}_{L^2(\Lambda_L)}\Big(\frac{\partial^n \mathcal{L}_L^{(\omega)}}{\partial b^n}(\beta, b, z, \epsilon)\Big) = \\ \left(\frac{q}{c}\right)^n \frac{\epsilon}{\beta|\Lambda_L|} \frac{i}{2\pi} \mathrm{Tr}_{L^2(\Lambda_L)}\Big(\int_{\Gamma_K} \mathrm{d}\xi\, \mathfrak{f}_\epsilon(\beta, z; \xi) \frac{\partial^n R_L}{\partial b^n}(b, \omega, \xi)\Big). \quad (4.5)$$

*Proof.* Let $\eta := \min\{E_0 - E_K, \frac{\vartheta_K}{2\beta}\} > 0$. From (2.7) and (2.5), $\mathbb{P}$-a.s., $\forall (b, b_0) \in \mathbb{R}^2$, $\forall 0 < L < \infty$ and $\forall \xi \in \mathbb{C}$, $d(\xi) \geq \eta$:

$$R_L(b, \omega, \xi) = R_L(b_0, \omega, \xi) + \sum_{k=1}^{2} \frac{(\delta b)^k}{k!} \frac{\partial^k R_L}{\partial b^k}(b_0, \omega, \xi) + \mathcal{S}_{3,L}(b, b_0, \omega, \xi). \quad (4.6)$$

(4.6) holds in the bounded operators sense. Under the conditions of Proposition 4.1 and in view of (4.1), we get from (4.3) followed by (4.6):

$$\mathcal{L}_L^{(\omega)}(\beta, b, z, \epsilon) = \sum_{k=0}^{2} \frac{(\delta b)^k}{k!} \Big\{\frac{i}{2\pi} \int_{\Gamma_K} \mathrm{d}\xi\, \mathfrak{f}_\epsilon(\beta, z; \xi) \frac{\partial^k R_L}{\partial b^k}(b_0, \omega, \xi)\Big\} + \\ \frac{i}{2\pi} \int_{\Gamma_K} \mathrm{d}\xi\, \mathfrak{f}_\epsilon(\beta, z; \xi) \mathcal{S}_{3,L}(b, b_0, \omega, \xi). \quad (4.7)$$

Now from (2.2), (2.3), (2.5) and the estimate $\|R_L(b_0,\omega,\xi)\| \leq \eta^{-1}$, $\xi \in \Gamma_K$, then $\mathbb{P}$-a.s., $\forall 0 < L < \infty$ there exists a polynomial $p(\cdot)$ s.t. $\forall b_0 \in \mathbb{R}$, $\forall \xi \in \Gamma_K$ and $k = 0,1,2$, $\|\frac{\partial^k R_L}{\partial b^k}(b_0,\omega,\xi)\| \leq |p(\xi)|$. Hence from (4.2), all the operators in the sum of the r.h.s. of (4.7) are bounded operators. Moreover in view of (2.8) with $n=2$, and since $b \in \mathbb{R}$, the same arguments as above applied to the last term of the r.h.s. of (4.7), show that this term behaves like $\mathcal{O}(|\delta b|^3)$ in the $\mathfrak{B}(L^2(\Lambda_L))$-sense. The proposition follows since $\mathbb{P}$-a.s., $\forall \beta > 0$, $\forall z \in K$ and $\forall 0 < L < \infty$, $\mathcal{L}_L^{(\omega)}(\beta,\cdot,z,\epsilon)$ is an $\mathfrak{I}_1(L^2(\Lambda_L))$-real analytic operator-valued function, see [6, proof of Prop. 3.5]. ■

We now give the main result of this section:

**Theorem 4.2.** *$\mathbb{P}$-a.s. on $\Omega$, $\forall 0 < L < \infty$, $\forall \beta > 0$, $\forall b \in \mathbb{R}$, $\forall z \in \mathcal{D}_\epsilon$ and for any compact subset $K$ of $\mathcal{D}_\epsilon$ s.t. $z \in K$, we have for $n = 1,2$:*

$$P_L^{(\omega)}(\beta,b,z,\epsilon) = \frac{\epsilon}{\beta|\Lambda_L|}\frac{i}{2\pi}\int_{\Lambda_L} \mathrm{d}\mathbf{x}\left(\int_{\Gamma_K} \mathrm{d}\xi\, \mathfrak{f}_\epsilon(\beta,z;\xi) R_L^{(1)}(\mathbf{x},\mathbf{y};b,\omega,\xi)\right)\Big|_{\mathbf{y}=\mathbf{x}}, \tag{4.8}$$

$$\mathcal{X}_{L,n}^{(\omega)}(\beta,b,z,\epsilon) = \left(\frac{q}{c}\right)^n \frac{n!\epsilon}{\beta|\Lambda_L|}\frac{i}{2\pi}\int_{\Gamma_K} \mathrm{d}\xi\, \mathfrak{f}_\epsilon(\beta,z;\xi)\int_{\Lambda_L} \mathrm{d}\mathbf{x} \sum_{k=1}^{n} \mathcal{T}_{k,L}^{n-k}(\mathbf{x},\mathbf{x};b,\omega,\xi), \tag{4.9}$$

*where $\mathcal{T}_{k,L}^{m}(\cdot\,,\cdot\,;b,\omega,\xi)$ with $k \in \{1,2\}$, $m \in \{0,1\}$ are given in (3.6).*

The above result together with the joint continuity of $\mathcal{T}_{k,L}^{m}(\cdot\,,\cdot\,;b,\omega,\xi)$ on $\Lambda_L^2$ lead to:

**Corollary 4.3.** *Under the same conditions as in Theorem 4.2, we have:*

$$\mathcal{X}_{L,1}^{(\omega)}(\beta,b,z,\epsilon) = -\left(\frac{q}{c}\right)\frac{\epsilon}{\beta|\Lambda_L|}\frac{i}{2\pi}\int_{\Gamma_K} \mathrm{d}\xi\, \mathfrak{f}_\epsilon(\beta,z;\xi)\mathrm{Tr}_{L^2(\Lambda_L)}\Big(R_L(b,\omega,\xi)T_{1,L}(b,\omega,\xi)\Big), \tag{4.10}$$

$$\begin{aligned}\mathcal{X}_{L,2}^{(\omega)}(\beta,b,z,\epsilon) = \left(\frac{q}{c}\right)^2 \frac{\epsilon}{\beta|\Lambda_L|}\frac{i}{\pi}\int_{\Gamma_K} \mathrm{d}\xi\, \mathfrak{f}_\epsilon(\beta,z;\xi)\times \\ \times \mathrm{Tr}_{L^2(\Lambda_L)}\bigg(R_L(b,\omega,\xi)\Big(\big(T_{1,L}(b,\omega,\xi)\big)^2 - T_{2,L}(b,\omega,\xi)\Big)\bigg),\end{aligned} \tag{4.11}$$

*where $T_{j,L}(b,\omega,\xi)$, $j=1,2$ are defined via their kernel in* (3.2) *and* (3.3) *respectively.*

*Proof of Theorem 4.2.* Under the conditions of Proposition 4.1 and for a fixed $\xi_0 < \min\{0,E_0\}$ and large enough, the first resolvent equation followed by the Cauchy integral formula lead to:

$$\mathcal{L}_L^{(\omega)}(\beta,b,z,\epsilon) = \frac{i}{2\pi}\left(\int_{\Gamma_K} \mathrm{d}\xi\,(\xi-\xi_0)\mathfrak{f}_\epsilon(\beta,z;\xi)R_L(b,\omega,\xi)\right)R_L(b,\omega,\xi_0).$$

From Remark 2.2 and in view of (4.1), then $\mathbb{P}$-a.s., $\forall b \in \mathbb{R}$, $\forall 0 < L < \infty$ and $\forall \xi \in \Gamma_K$, the operator $R_L(b,\omega,\xi)R_L(b,\omega,\xi_0)$ has a jointly continuous kernel. Moreover $\mathbb{P}$-a.s., there exists a polynomial $p(\cdot)$ s.t. $\forall b \in \mathbb{R}$, $\forall \xi \in \Gamma_K$ and $\forall (\mathbf{x},\mathbf{y}) \in \Lambda_L^2$, $|(R_L(b,\omega,\xi)R_L(b,\omega,\xi_0))(\mathbf{x},\mathbf{y})| \leq |p(\xi)|$. By using (4.2), the joint continuity of the integral kernel of $\mathcal{L}_L^{(\omega)}(\beta,b,z,\epsilon)$ follows by standard arguments. This proves (4.8). Let $n = 1,2$. Clearly as in the proof of Proposition 4.1, we have for any $\varphi \in \mathcal{C}_0^\infty(\Lambda_L)$ and $\mathbf{x} \in \Lambda_L$:

$$\Big(\frac{\partial^n \mathcal{L}_L^{(\omega)}}{\partial b^n}(\beta,b,z,\epsilon)\varphi\Big)(\mathbf{x}) = \frac{i}{2\pi}\int_{\Gamma_K} \mathrm{d}\xi\, \mathfrak{f}_\epsilon(\beta,z;\xi)\int_{\Lambda_L} \mathrm{d}\mathbf{y}\, \frac{\partial^n R_L^{(1)}}{\partial b^n}(\mathbf{x},\mathbf{y};b,\omega,\xi)\varphi(\mathbf{y}). \tag{4.12}$$

The estimates (2.14), (2.15), (4.2) and standard arguments then imply:

$$\forall(\mathbf{x},\mathbf{y})\in\Lambda_L^2,\quad \frac{\partial^n\mathcal{L}_L^{(\omega)}}{\partial b^n}(\mathbf{x},\mathbf{y};\beta,b,z,\epsilon)=\frac{i}{2\pi}\int_{\Gamma_K}\mathrm{d}\xi\,\mathfrak{f}_\epsilon(\beta,z;\xi)\frac{\partial^n R_L^{(1)}}{\partial b^n}(\mathbf{x},\mathbf{y};b,\omega,\xi).\tag{4.13}$$

Now use (3.8). Then from (4.13), we have on $\Lambda_L^2$:

$$\begin{aligned}\frac{\partial^n\mathcal{L}_L^{(\omega)}}{\partial b^n}(\mathbf{x},\mathbf{y};\beta,b,z,\epsilon)=n!\sum_{k=1}^n\frac{i}{2\pi}\int_{\Gamma_K}\mathrm{d}\xi\,\mathfrak{f}_\epsilon(\beta,z;\xi)\mathcal{T}_{k,L}^{n-k}(\mathbf{x},\mathbf{y};b,\omega,\xi)+\\+(i\phi(\mathbf{x},\mathbf{y}))^n\mathcal{L}_L^{(\omega)}(\mathbf{x},\mathbf{y};\beta,b,z,\epsilon).\end{aligned}\tag{4.14}$$

We have already proved that the second term in the r.h.s. of (4.14) is jointly continuous on $\Lambda_L^2$. Besides we know from Section 3 that $\mathbb{P}$-a.s., $\forall b\in\mathbb{R}$, $\forall 0<L<\infty$, $\forall\xi\in\Gamma_K$, $\mathcal{T}_{k,L}^m(\cdot\,,\cdot\,;b,\omega,\xi)$, $k=1,2$, $m=0,1$, are jointly continuous on $\Lambda_L^2$. Moreover we have the bound (3.7). Again by standard arguments, we conclude that each term in the sum of the r.h.s. of (4.14) is jointly continuous on $\Lambda_L^2$. So we perform the trace in (4.5) as the integral on $\Lambda_L$ of the diagonal part of the integral kernel (4.14). This together with $\phi(\mathbf{x},\mathbf{x})=0$ imply (4.9). ■

**Remark 4.4.** Due to (2.5), (4.5) can be extended for any integer $n\geq 3$ to define the generalized susceptibilities at finite volume, see [3, 4, 5]. If $n=3$:

$$\begin{aligned}&\mathcal{X}_{L,3}^{(\omega)}(\beta,b,z,\epsilon):=\left(\frac{q}{c}\right)^3\frac{\partial^3P_L^{(\omega)}}{\partial b^3}(\beta,b,z,\epsilon)=\\&\left(\frac{q}{c}\right)^3\frac{\epsilon}{\beta|\Lambda_L|}\frac{3i}{\pi}\int_{\Gamma_K}\mathrm{d}\xi\,\mathfrak{f}_\epsilon(\beta,z;\xi)\left\{\int_{\Lambda_L}\mathrm{d}\mathbf{x}\,\mathcal{U}_{L,1}^{(\omega)}(\mathbf{x},\mathbf{x};b,\xi)+\int_{\Lambda_L}\mathrm{d}\mathbf{x}\,\mathcal{U}_{L,2}^{(\omega)}(\mathbf{x},\mathbf{x};b,\xi)\right\},\end{aligned}\tag{4.15}$$

where: $\mathcal{U}_{L,1}^{(\omega)}(\mathbf{x},\mathbf{y};b,\xi):=$

$$\Big(R_L(b,\omega,\xi)\Big(T_{1,L}(b,\omega,\xi)T_{2,L}(b,\omega,\xi)+T_{2,L}(b,\omega,\xi)T_{1,L}(b,\omega,\xi)-\big(T_{1,L}(b,\omega,\xi)\big)^3\Big)\Big)(\mathbf{x},\mathbf{y}),$$

$$\begin{aligned}\mathcal{U}_{L,2}^{(\omega)}(\mathbf{x},\mathbf{y};b,\xi):=\int_{\Lambda_L}\mathrm{d}\mathbf{z}_1\int_{\Lambda_L}\mathrm{d}\mathbf{z}_2\int_{\Lambda_L}\mathrm{d}\mathbf{z}_3\,\big(i(\phi(\mathbf{x},\mathbf{z}_1)+\phi(\mathbf{z}_1,\mathbf{z}_2)+\phi(\mathbf{z}_2,\mathbf{y}))\big)\times\\ R_L^{(1)}(\mathbf{x},\mathbf{z}_1;b,\omega,\xi)T_{1,L}(\mathbf{z}_1,\mathbf{z}_2;b,\omega,\xi)T_{1,L}(\mathbf{z}_2,\mathbf{y};b,\omega,\xi).\end{aligned}\tag{4.16}$$

Due to (2.9), (3.4) and the estimate $|\phi(\mathbf{x},\mathbf{z}_1)+\phi(\mathbf{z}_1,\mathbf{z}_2)+\phi(\mathbf{z}_2,\mathbf{x})|\leq|\mathbf{x}-\mathbf{z}_1||\mathbf{z}_1-\mathbf{z}_2|$ (see (3.1)), $\mathbb{P}$-a.s., $\forall b\in\mathbb{R}$, $\forall 0<L<\infty$, $\forall\xi\in\Gamma_K$, the integral kernels $\mathcal{U}_{L,j}^{(\omega)}(\cdot\,,\cdot\,;b,\xi)$, $j=1,2$ are well-defined on $\Lambda_L^2$ and eventually jointly continuous.

## 5. The bulk operators

5.1. **Preliminaries.** In the following we denote by $P_l(b):=\mathbf{P}(b)\cdot\mathbf{e}_l$, $l=1,2$ where $\mathbf{P}(b):=(i\nabla_{\mathbf{x}}+b\mathbf{a}(\mathbf{x}))$, $\mathbf{e}_1:=(1,0,0)$ and $\mathbf{e}_2:=(0,1,0)$.

**Lemma 5.1.** *$\mathbb{P}$-a.s. on $\Omega$, $\forall b\in\mathbb{R}$, $\forall\eta>0$, $\forall\xi\in\mathbb{C}$, $d(\xi)\geq\eta$, then $P_k(b)R_\infty(b,\omega,\xi)P_l(b)$, $k,l=1,2$ are bounded operators and there exists a polynomial $p(\cdot)$ independent of $(\omega,b)$ s.t.:*

$$\|P_k(b)R_\infty(b,\omega,\xi)P_l(b)\|\leq|p(\xi)|.\tag{5.1}$$

*Proof.* By using (1.20), the analysis of [23, Sect. A2] and the diamagnetic inequality, we know that for any $\varepsilon > 0$, then if $\xi < E_0$ and large enough, $\mathbb{P}$-a.s., $\forall b \in \mathbb{R}$:

$$\| |V_1^{(\omega)}|^{1/2} R_\infty(b,\omega,\xi) |V_1^{(\omega)}|^{1/2} \| \leq \varepsilon.$$

This implies that $\forall \varepsilon > 0$, there exists $a(\varepsilon)$ independent of $\omega$ and $b$ s.t. $\forall \varphi \in D(H_\infty(b,\omega))$:

$$(\varphi, |V_1^{(\omega)}|\varphi) \leq \varepsilon |(\varphi, H_\infty(b,\omega)\varphi)| + a(\varepsilon)\|\varphi\|_2^2. \tag{5.2}$$

Besides $\mathbb{P}$-a.s., $\forall b \in \mathbb{R}$, $D(H_\infty(b,\omega)) \subset \{\phi \in L^2(\mathbb{R}^3) : (i\nabla + b\mathbf{a})\phi \in (L^2(\mathbb{R}^3))^3,\ (V_2^{(\omega)})^{\frac{1}{2}}\phi \in L^2(\mathbb{R}^3)\}$, see [23, Sect. B13]. Under these conditions we conclude the following. From (5.2), for any $\varphi \in D(H_\infty(b,\omega))$:

$$\frac{1}{2}\|P_k(b)\varphi\|_2^2 \leq (1+\varepsilon)\Re(\varphi, (H_\infty(b,\omega) - \xi)\varphi) + ((1+\varepsilon)|\Re\xi| + a(\varepsilon))\|\varphi\|_2^2. \tag{5.3}$$

Choose $\varphi = R_\infty(b,\omega,\xi)\psi$ with $\psi \in L^2(\mathbb{R}^3)$, $\|\psi\|_2 = 1$ and $\xi \in \mathbb{C}$, $d(\xi) \geq \eta$, then (5.3) shows that there exists a constant $c_1 > 0$ independent of $\omega$ and $b$ s.t.:

$$\|P_k(b)R_\infty(b,\omega,\xi)\psi\|_2 \leq c_1(1+|\xi|)^{1/2}. \tag{5.4}$$

Therefore $P_k(b)R_\infty(b,\omega,\xi)$ is a bounded operator and $\|P_k(b)R_\infty(b,\omega,\xi)\| \leq c_1(1+|\xi|)^{1/2}$. Now choose $\varphi = R_\infty(b,\omega,\xi)P_k(b)\psi$ with $\psi \in D(H_\infty(b,\omega))$, $\|\psi\|_2 = 1$, and $\xi \in \mathbb{C}$, $d(\xi) \geq \eta$, then (5.3) and (5.4) imply that there exists a constant $c_2 > 0$ independent of $(\omega, b)$ s.t.:

$$\|P_k(b)R_\infty(b,\omega,\xi)P_k(b)\psi\|_2^2 \leq c_2(\|P_k(b)R_\infty(b,\omega,\xi)P_k(b)\psi\|_2 + (1+|\xi|)^2). \tag{5.5}$$

Hence $P_k(b)R_\infty(b,\omega,\xi)P_k(b)$ is bounded and $\|P_k(b)R_\infty(b,\omega,\xi)P_k(b)\| \leq c_2'(1+|\xi|)$ for another constant $c_2' > 0$ independent of $(\omega, b)$. Let $\varphi = R_\infty(b,\omega,\xi)P_l(b)\psi$, with $\psi \in D(H_\infty(b,\omega))$, $\|\psi\|_2 = 1$, then (5.3)-(5.5) together imply:

$$\|P_k(b)R_\infty(b,\omega,\xi)P_l(b)\psi\|_2^2 \leq c_2(\|P_l(b)R_\infty(b,\omega,\xi)P_l(b)\psi\|_2 + (1+|\xi|)^2).$$

Again $P_k(b)R_\infty(b,\omega,\xi)P_l(b), k \neq l$ is bounded and $\|P_k(b)R_\infty(b,\omega,\xi)P_l(b)\| \leq c_3(1+|\xi|)$ for some constant $c_3 > 0$ independent of $(\omega, b)$. ■

Let $\{\mathbf{T}_{\mathbf{k},b}\}_{\mathbf{k}\in\mathbb{R}^3}$ be the family of the usual real magnetic translations defined as the following. Let $\phi$ be the phase defined as in (3.1), and:

$$\forall \mathbf{k} \in \mathbb{R}^3, \quad (\mathbf{T}_{\mathbf{k},b}\psi)(\mathbf{x}) := e^{ib\phi(\mathbf{x},\mathbf{k})}\psi(\mathbf{x}-\mathbf{k}) \quad \psi \in L^2(\mathbb{R}^3). \tag{5.6}$$

**Lemma 5.2.** *$\mathbb{P}$-a.s. on $\Omega$, $\forall b \in \mathbb{R}$, $\forall \eta > 0$ and $\forall \xi \in \mathbb{C}$, $d(\xi) \geq \eta$, then for $j = 1, 2$:*
*i) $T_{j,\infty}(b,\omega,\xi)$ is a bounded operator and there exists a polynomial $p(\cdot)$ independent of $(\omega, b)$ s.t.:*

$$\|T_{j,\infty}(b,\omega,\xi)\| \leq |p(\xi)|. \tag{5.7}$$

*ii) $T_{j,\infty}(b,\omega,\xi)$ satisfies the covariance relation:*

$$\forall \mathbf{k} \in \mathbb{R}^3, \quad \mathbf{T}_{\mathbf{k},b}T_{j,\infty}(b,\omega,\xi)\mathbf{T}_{-\mathbf{k},b} = T_{j,\infty}(b,\tau_{\mathbf{k}}\omega,\xi). \tag{5.8}$$

*Here $T_{j,\infty}(b,\omega,\xi)$, $j = 1, 2$ are defined via their integral kernel in* (1.12), (1.13) *respectively.*

*Proof.* Let us consider the operator $T_{1,\infty}(b,\omega,\xi)$. In view of its integral kernel (1.12), the definition of the symmetric gauge and under the conditions of Lemma 5.2, we have on $\mathbb{R}^6 \setminus D_\infty$:

$$T_{1,\infty}(\mathbf{x},\mathbf{y};b,\omega,\xi) = \frac{1}{2}(i\nabla_{\mathbf{x}} + b\mathbf{a}(\mathbf{x})) \cdot (-(x_2 - y_2)\mathbf{e}_1 + (x_1 - y_1)\mathbf{e}_2)R_\infty^{(1)}(\mathbf{x},\mathbf{y};b,\omega,\xi).$$

By using the same arguments as the ones in the proof of [7, Prop. 3.2], we get:

$$T_{1,\infty}(b,\omega,\xi) = \frac{i}{2}\Big(P_1(b)R_\infty(b,\omega,\xi)P_2(b) - P_2(b)R_\infty(b,\omega,\xi)P_1(b)\Big)R_\infty(b,\omega,\xi), \tag{5.9}$$

which is valid in the form sense on $\mathcal{C}_0^\infty(\mathbb{R}^3)\times\mathcal{C}_0^\infty(\mathbb{R}^3)$. By Lemma 5.1, $\mathbb{P}$-a.s., $\forall b\in\mathbb{R}$, $\forall\eta>0$, $\forall\xi\in\mathbb{C}$, $d(\xi)\geq\eta$, the r.h.s of (5.9) defines a bounded operator on $L^2(\mathbb{R}^3)$. So this holds for the operator $T_{1,\infty}(b,\omega,\xi)$. The same arguments can be applied to the operator $T_{2,\infty}(b,\omega,\xi)$, but from the equality valid in the form sense on $L^2(\mathbb{R}^3)$:

$$\begin{aligned} T_{2,\infty}(b,\omega,\xi) = -\frac{1}{4}\Big(&R_\infty(b,\omega,\xi)P_1(b)R_\infty(b,\omega,\xi)P_1(b)+ \\ &+R_\infty(b,\omega,\xi)P_2(b)R_\infty(b,\omega,\xi)P_2(b) - R_\infty(b,\omega,\xi)\Big)R_\infty(b,\omega,\xi). \end{aligned} \tag{5.10}$$

This proves *i*). Let us show *ii*). The measurability of $\omega\mapsto H_\infty(b,\omega)$ combined with the assumption (E) lead to the covariance relation:

$$\forall\mathbf{k}\in\mathbb{R}^3,\quad \mathbf{T}_{\mathbf{k},b}R_\infty(b,\omega,\xi)\mathbf{T}_{-\mathbf{k},b} = R_\infty(b,\tau_{\mathbf{k}}\omega,\xi). \tag{5.11}$$

This implies the following identity on $\mathbb{R}^6\setminus D_\infty$:

$$\forall\mathbf{k}\in\mathbb{R}^3,\quad R_\infty^{(1)}(\mathbf{x}-\mathbf{k},\mathbf{y}-\mathbf{k};b,\omega,\xi)\mathrm{e}^{-ib\phi(\mathbf{y},\mathbf{k})} = \mathrm{e}^{-ib\phi(\mathbf{x},\mathbf{k})}R_\infty^{(1)}(\mathbf{x},\mathbf{y};b,\tau_{\mathbf{k}}\omega,\xi),$$

and from (3.17), $\mathrm{e}^{-ib\phi(\mathbf{x},\mathbf{k})}\mathbf{a}(\mathbf{x}-\mathbf{y})(i\nabla_{\mathbf{x}}+b\mathbf{a}(\mathbf{x})) = \mathbf{a}(\mathbf{x}-\mathbf{y})(i\nabla_{\mathbf{x}}+b\mathbf{a}(\mathbf{x}-\mathbf{k}))\mathrm{e}^{-ib\phi(\mathbf{x},\mathbf{k})}$. Then (5.8) follows from these two relations together with (1.12)-(1.13). ■

We now use these results to investigate the $\mathbb{P}$-a.s. properties of the operators involved in the definition (1.15) and (1.16). Introduce the notation:

$$\mathcal{I}_j(b,\omega,\xi) := R_\infty(b,\omega,\xi)T_{j,\infty}(b,\omega,\xi)\quad j\in\{1,2\}, \tag{5.12}$$

$$\mathcal{I}_3(b,\omega,\xi) := R_\infty(b,\omega,\xi)T_{1,\infty}^2(b,\omega,\xi), \tag{5.13}$$

and also set:

$$\mathcal{I}_0(b,\omega,\xi) = \mathcal{I}_0(b,\omega,\xi,\xi_0) := R_\infty(b,\omega,\xi)R_\infty^2(b,\omega,\xi_0), \tag{5.14}$$

for a fixed $\xi_0<\min\{0,E_0\}$ (see (1.7)) and large enough. Notice that from Lemma 5.2, $\mathbb{P}$-a.s., $\forall b\in\mathbb{R}$, $\forall\eta>0$ and $\forall\xi\in\mathbb{C}$, $d(\xi)\geq\eta$, these operators are bounded on $L^2(\mathbb{R}^3)$. Below we denote by $\chi_U$ the characteristic function of a given $U\subset\mathbb{R}^3$.

**Proposition 5.3.** *$\mathbb{P}$-a.s. on $\Omega$, $\forall b\in\mathbb{R}$, $\forall\eta>0$, $\forall\xi\in\mathbb{C}$, $d(\xi)\geq\eta$, one has for $j=0,1,2,3$:*
*i) For any open and bounded set $U\subset\mathbb{R}^3$, $\chi_U\mathcal{I}_j(b,\omega,\xi)\chi_U$ is trace class on $L^2(\mathbb{R}^3)$ and*

$$\|\chi_U\mathcal{I}_j(b,\omega,\xi)\chi_U\|_{\mathfrak{I}_1}\leq|p(\xi)|,$$

*for some polynomial $p(\cdot)$.*
*ii) $\mathcal{I}_j(b,\omega,\xi)$ has an integral kernel $\mathcal{I}_j(\cdot\,,\cdot;b,\omega,\xi)$ jointly continuous on $\mathbb{R}^6$, and moreover there exists a polynomial $p(\cdot)$ independent of $(\omega,b)$ s.t.:*

$$\forall(\mathbf{x},\mathbf{y})\in\mathbb{R}^6,\quad |\mathcal{I}_j(\mathbf{x},\mathbf{y};b,\omega,\xi)|\leq|p(\xi)|. \tag{5.15}$$

*iii) $(\omega,\mathbf{x})\mapsto\mathcal{I}_j(\mathbf{x},\mathbf{x};b,\omega,\xi)$ is a $\mathbb{R}^3$-ergodic random field with a finite expectation:*

$$\forall\mathbf{x}\in\mathbb{R}^3,\quad \mathbb{E}[|\mathcal{I}_j(\mathbf{x},\mathbf{x};b,\omega,\xi)|]<\infty. \tag{5.16}$$

*Proof.* From (2.9) and (2.11), $\mathbb{P}$-a.s., $\forall b \in \mathbb{R}$, $\forall \eta > 0$ there exists $\gamma = \gamma(\eta) > 0$ and a polynomial $p(\cdot)$ s.t. $\forall L \in (0, \infty]$ and $\forall \xi \in \mathbb{C}$, $d(\xi) \geq \eta$, on $\Lambda_L^2 \setminus D_L$:

$$|R_L^{(1)}(\mathbf{x}, \mathbf{y}; b, \omega, \xi)|, |T_{2,L}(\mathbf{x}, \mathbf{y}; b, \omega, \xi)| \leq |p(\xi)| \frac{\mathrm{e}^{-\frac{\gamma}{1+|\xi|}|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|}, \tag{5.17}$$

$$|T_{1,L}(\mathbf{x}, \mathbf{y}; b, \omega, \xi)| \leq |p(\xi)|(1 + |\mathbf{x}|^\alpha + |\mathbf{y}|^\alpha) \frac{\mathrm{e}^{-\frac{\gamma}{1+|\xi|}|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|}. \tag{5.18}$$

Then we have the following estimates [16]:

$$\|\chi_U R_\infty(b, \omega, \xi)\|_{\mathfrak{I}_2}, \|\chi_U T_{1,\infty}(b, \omega, \xi)\|_{\mathfrak{I}_2}, \ \|\chi_U T_{2,\infty}(b, \omega, \xi)\|_{\mathfrak{I}_2} \leq |p(\xi)|,$$

for some polynomial $p(\cdot)$. Besides we have (5.7). By standard operator estimates, $i)$ follows. Let us show $ii)$. From (5.17), (5.18) and Lemma 7.1 $ii)$, the integral kernels of $\mathcal{I}_j(b, \omega, \xi)$, $j = 0, 1, 2$ are jointly continuous on $\mathbb{R}^6$. This also holds for $\mathcal{I}_3(\cdot\,, \cdot\,; b, \omega, \xi)$ from the identity:

$$\mathcal{I}_3(\mathbf{x}, \mathbf{y}; b, \omega, \xi) = \int_{\mathbb{R}^3} \mathrm{d}\mathbf{z}\, \mathcal{I}_1(\mathbf{x}, \mathbf{z}; b, \omega, \xi) T_{1,\infty}(\mathbf{z}, \mathbf{y}; b, \omega, \xi),$$

followed by Lemma 7.1 $ii)$ together with (5.17) and the rough estimate:

$$\forall\, (\mathbf{x}, \mathbf{y}) \in \mathbb{R}^6, \quad |\mathcal{I}_1(\mathbf{x}, \mathbf{y}; b, \omega, \xi)| \leq |p(\xi)|(1 + |\mathbf{x}|^\alpha + |\mathbf{y}|^\alpha),$$

for some polynomial $p(\cdot)$. We now prove (5.15). Although the case of $j = 0$ is straightforward, the method given below covers all cases $j = 0, 1, 2, 3$. Let $\xi_0 < \min\{0, E_0\}$. We firstly show:

$$\mathcal{I}_j(b, \omega, \xi) = R_\infty(b, \omega, \xi_0) A_j(b, \omega, \xi) R_\infty(b, \omega, \xi_0), \tag{5.19}$$

where $A_j(b, \omega, \xi)$ consists of a finite linear combination of operator products of type:

$$(\xi - \xi_0)^q R_\infty^r(b, \omega, \xi) \{P_k(b) R_\infty(b, \omega, \xi) P_l(b)\}^s R_\infty^t(b, \omega, \xi) \quad r + s + t \geq 1, \tag{5.20}$$

where the exponents $q, r, t \in \{0, 1, 2\}$, $s \in \{0, 1\}$, $k, l \in \{1, 2\}$ depend on $j$. Notice that Lemma 5.1 implies that $\mathbb{P}$-a.s., $\forall \eta > 0$ there exists a $\omega$-independent polynomial $p(\cdot)$ s.t. $\forall b \in \mathbb{R}$ and $\forall \xi \in \mathbb{C}$, $d(\xi) \geq \eta$:

$$\|A_j(b, \omega, \xi)\| \leq |p(\xi)|. \tag{5.21}$$

Obviously (5.19) holds for $j = 0$. If $j = 1, 2$, we use once the first resolvent equation for the resolvent appearing in (5.12):

$$\mathcal{I}_j(b, \omega, \xi) = R_\infty(b, \omega, \xi_0) T_{j,\infty}(b, \omega, \xi) + (\xi - \xi_0) R_\infty(b, \omega, \xi_0) \mathcal{I}_j(b, \omega, \xi).$$

Then we use (5.9)-(5.10), and again the first resolvent equation for the last resolvent in the expression (5.9)-(5.10). This leads to (5.19) by a straightforward calculation. For $j = 3$, we use twice the identity (5.9) in (5.13) and we repeat the same procedure as above. This proves (5.19). Furthermore from (1.20), [23, Sect.A2] together with [8, Eq. (2.40)], then $\mathbb{P}$-a.s., $\forall \xi_0 < \min\{0, E_0\}$ and large enough, there exists a $\omega$-independent constant $c > 0$ s.t. $\forall b \in \mathbb{R}$:

$$\|R_\infty(b, \omega, \xi_0)\|_{1,2} = \|R_\infty(b, \omega, \xi_0)\|_{2,\infty} \leq c,$$

where $\|\cdot\|_{p,q}$ denotes the norm for operator from $L^p(\mathbb{R}^3)$ to $L^q(\mathbb{R}^3)$, $1 \leq p, q \leq \infty$. Further, let $B(b, \omega, \xi)$ be a bounded operator on $L^2(\mathbb{R}^3)$. Then for any $\varphi, \psi \in \mathcal{C}_0^\infty(\mathbb{R}^3)$:

$$|\big(\varphi, R_\infty(b, \omega, \xi_0) B(b, \omega, \xi) R_\infty(b, \omega, \xi_0)\psi\big)| \leq c^2 \|\varphi\|_1 \|\psi\|_1 \|B(b, \omega, \xi)\|.$$

Suppose that the operator $R_\infty(b, \omega, \xi_0) B(b, \omega, \xi) R_\infty(b, \omega, \xi_0)$ has a jointly continuous integral kernel. Then by using a limiting procedure we conclude that:

$$\forall (\mathbf{x}, \mathbf{y}) \in \mathbb{R}^6, \quad |(R_\infty(b, \omega, \xi_0) B(b, \omega, \xi) R_\infty(b, \omega, \xi_0))(\mathbf{x}, \mathbf{y})| \leq c^2 \|B(b, \omega, \xi)\|. \tag{5.22}$$

Thus by setting $B(b,\omega,\xi) = A_j(b,\omega,\xi)$, we get (5.15) from (5.19) and (5.21).
Let us prove *iii*). As a result of (5.11) and (5.8), the following covariance relation holds: $\forall \mathbf{k} \in \mathbb{R}^3$, $T_{\mathbf{k},b}\mathcal{I}_j(b,\omega,\xi)T_{-\mathbf{k},b} = \mathcal{I}_j(b,\tau_{\mathbf{k}}\omega,\xi)$. This implies $\forall(\mathbf{x},\mathbf{y}) \in \mathbb{R}^6$:

$$\forall \mathbf{k} \in \mathbb{R}^3, \quad \mathcal{I}_j(\mathbf{x},\mathbf{y};b,\tau_{\mathbf{k}}\omega,\xi) = \mathrm{e}^{ib\phi(\mathbf{x},\mathbf{k})}\mathcal{I}_j(\mathbf{x}-\mathbf{k},\mathbf{y}-\mathbf{k};b,\omega,\xi)\mathrm{e}^{-ib\phi(\mathbf{y},\mathbf{k})}.$$

Then $(\omega,\mathbf{x}) \mapsto \mathcal{I}_j(\mathbf{x},\mathbf{x};b,\omega,\xi)$ is well defined and $\mathbb{R}^3$-stationary. (5.16) follows from the $\omega$-independent estimate (5.15). ■

We deduce properties of the operators in (1.14)-(1.16) from the following. Let $\xi_0$ as above. The first resolvent equation and the Cauchy integral formula imply that $\mathbb{P}$ a.s., $\forall\beta > 0$, $\forall b \in \mathbb{R}$, $\forall z \in \mathcal{D}_\epsilon$ and $\forall K$ compact subset of $\mathcal{D}_\epsilon$, s.t. $z \in K$:

$$\mathcal{L}_{\infty,0}^{(\omega)}(\beta,b,z,\epsilon) = \frac{i}{2\pi}\int_{\Gamma_K} \mathrm{d}\xi\, (\xi-\xi_0)^2 \mathfrak{f}_\epsilon(\beta,z;\xi)\mathcal{I}_0(b,\omega,\xi), \tag{5.23}$$

and, from (5.12)-(5.13),

$$\mathcal{L}_{\infty,1}^{(\omega)}(\beta,b,z,\epsilon) = -\frac{i}{2\pi}\int_{\Gamma_K} \mathrm{d}\xi\, \mathfrak{f}_\epsilon(\beta,z;\xi)\mathcal{I}_1(b,\omega,\xi), \tag{5.24}$$

$$\mathcal{L}_{\infty,2}^{(\omega)}(\beta,b,z,\epsilon) = \frac{i}{\pi}\int_{\Gamma_K} \mathrm{d}\xi\, \mathfrak{f}_\epsilon(\beta,z;\xi)\Big(\mathcal{I}_3(b,\omega,\xi) - \mathcal{I}_2(b,\omega,\xi)\Big).$$

Then Proposition 5.3 together with the estimate (4.2) imply:

**Corollary 5.4.** *$\mathbb{P}$-a.s. on $\Omega$, $\forall\beta > 0$, $\forall b \in \mathbb{R}$, $\forall z \in \mathcal{D}_\epsilon$ and for any compact subset $K$ of $\mathcal{D}_\epsilon$ s.t. $z \in K$, one has for $n = 0,1,2$:*
*i) For all open and bounded set $U \subset \mathbb{R}^3$, $\chi_U\mathcal{L}_{\infty,n}^{(\omega)}(\beta,b,z,\epsilon)\chi_U$ is trace class on $L^2(\mathbb{R}^3)$.*
*ii) $\mathcal{L}_{\infty,n}^{(\omega)}(\beta,b,z,\epsilon)$ has an integral kernel $\mathcal{L}_{\infty,n}^{(\omega)}(\cdot\,,\cdot\,;\beta,b,z,\epsilon)$ jointly continuous on $\mathbb{R}^6$.*
*iii) $(\omega,\mathbf{x}) \mapsto \mathcal{L}_{\infty,n}^{(\omega)}(\mathbf{x},\mathbf{x};\beta,b,z,\epsilon)$ is a $\mathbb{R}^3$-ergodic random field with a finite expectation.*

The last two results of this section needed to prove our main theorems are the following.

**Proposition 5.5.** *$\mathbb{P}$-a.s. on $\Omega$, $\forall b \in \mathbb{R}$ and $j = 0,1,2,3$, then the following maps:*
*i)*

$$\xi \in \Gamma_K \mapsto \frac{1}{|\Lambda_L|}\mathrm{Tr}_{L^2(\mathbb{R}^3)}\Big(\chi_{\Lambda_L}\mathcal{I}_j(b,\omega,\xi)\chi_{\Lambda_L}\Big),$$

*is continuous uniformly in $L \in (0,\infty)$.*
*ii) $\xi \in \Gamma_K \mapsto \mathbb{E}[\mathcal{I}_j(\mathbf{0},\mathbf{0};b,\omega,\xi)]$ is continuous.*

*Proof.* Let $\xi_1 \in \Gamma_K$ be fixed. Due to Proposition 5.3, to prove *i*) it is sufficient to show that $\mathbb{P}$-a.s., $\forall b \in \mathbb{R}$, there exists a constant $c > 0$ independent of $L$ s.t. $\forall \mathbf{x} \in \mathbb{R}^3$, $\forall\xi \in \Gamma_K$ with $|\delta\xi| := |\xi - \xi_1|$ small enough:

$$|\mathcal{I}_j(\mathbf{x},\mathbf{x};b,\omega,\xi) - \mathcal{I}_j(\mathbf{x},\mathbf{x};b,\omega,\xi_1)| \leq c|\delta\xi|. \tag{5.25}$$

Let $j = 0$. From (5.14) together with the first resolvent equation, we get for any $\mathbf{x} \in \mathbb{R}^3$:

$$\mathcal{I}_0(\mathbf{x},\mathbf{x};b,\omega,\xi) - \mathcal{I}_0(\mathbf{x},\mathbf{x};b,\omega,\xi_1) = \\ (\xi-\xi_1)\big(R_\infty(b,\omega,\xi_0)R_\infty(b,\omega,\xi)R_\infty(b,\omega,\xi_1)R_\infty(b,\omega,\xi_0)\big)(\mathbf{x},\mathbf{x}).$$

Since $\|R_\infty(b,\omega,\xi)R_\infty(b,\omega,\xi_1)\| \leq \eta^{-2}$, by applying (5.22) with $B(b,\omega,\xi) = R_\infty(b,\omega,\xi)R_\infty(b,\omega,\xi_1)$, we get (5.25). Let $j = 1,2,3$. In view of (5.19), we have to estimate:

$$\mathcal{I}_j(\mathbf{x},\mathbf{x};b,\omega,\xi) - \mathcal{I}_j(\mathbf{x},\mathbf{x};b,\omega,\xi_1) = \\ \Big(R_\infty(b,\omega,\xi_0)\big(A_j(b,\omega,\xi) - A_j(b,\omega,\xi_1)\big)R_\infty(b,\omega,\xi_0)\Big)(\mathbf{x},\mathbf{x}), \quad (5.26)$$

where $A_j(b,\omega,\cdot)$ consists of a finite linear combination of operators of type (5.20). Choose a generic term appearing in $(A_j(b,\omega,\xi) - A_j(b,\omega,\xi_1))$:

$$C(b,\omega,\xi,\xi_1) := (\xi-\xi_0)^q R_\infty(b,\omega,\xi)P_k(b)R_\infty(b,\omega,\xi)P_l(b) + \\ -(\xi_1-\xi_0)^q R_\infty(b,\omega,\xi_1)P_k(b)R_\infty(b,\omega,\xi_1)P_l(b) \quad q \in \{0,1,2\}.$$

By using twice the resolvent equation in the first term of the r.h.s. of this formula, we get:

$$C(b,\omega,\xi,\xi_1) = ((\xi-\xi_0)^q - (\xi_1-\xi_0)^q)R_\infty(b,\omega,\xi_1)P_k(b)R_\infty(b,\omega,\xi_1)P_l(b) + (\xi-\xi_1)(\xi-\xi_0)^q \times \\ R_\infty(b,\omega,\xi_1)\Big(R_\infty(b,\omega,\xi)P_k(b)R_\infty(b,\omega,\xi)P_l(b) + P_k(b)R_\infty(b,\omega,\xi)R_\infty(b,\omega,\xi_1)P_l(b)\Big).$$

By (5.1) and (5.4), $\mathbb{P}$-a.s., $\forall b \in \mathbb{R}$, $\forall \xi \in \Gamma_K$ sufficiently near $\xi_1$, there exists $c > 0$ independent of $\xi$ and $L$ s.t. $\|C(b,\omega,\xi,\xi_1)\| \leq c|\delta\xi|$. This also holds for $\|A_j(b,\omega,\xi) - A_j(b,\omega,\xi_1)\|$ which implies (5.25) from (5.22) and (5.26). Now *ii)* follows from the continuity of $\xi \mapsto \mathcal{I}_j(\mathbf{x},\mathbf{x};b,\omega,\xi)$, $\forall \mathbf{x} \in \mathbb{R}^3$ together with the $\omega$-independent estimate (5.15). ■

**Proposition 5.6.** *$\mathbb{P}$-a.s. on $\Omega$, $\forall \xi \in \Gamma_K$ and $j = 0,1$, then the following maps:*
*i)*

$$b \in \mathbb{R} \mapsto \frac{1}{|\Lambda_L|}\mathrm{Tr}_{L^2(\mathbb{R}^3)}\Big(\chi_{\Lambda_L}\mathcal{I}_j(b,\omega,\xi)\chi_{\Lambda_L}\Big),$$

*is continuous uniformly in $L \in (0,\infty)$.*
*ii) $b \in \mathbb{R} \mapsto \mathbb{E}[\mathcal{I}_j(\mathbf{0},\mathbf{0};b,\omega,\xi)]$ is continuous.*

To prove this result we need the following. Introduce on $L^2(\mathbb{R}^3)$ the operator $\mathbf{W}(b,b_0,\omega,\xi)$ through its integral kernel defined on $\mathbb{R}^6 \setminus D_\infty$ as:

$$\mathbf{W}(\mathbf{x},\mathbf{y};b,b_0,\omega,\xi) := \mathrm{e}^{i\delta b\phi(\mathbf{x},\mathbf{y})}\mathbf{a}(\mathbf{x}-\mathbf{y})R_\infty^{(1)}(\mathbf{x},\mathbf{y};b_0,\omega,\xi). \quad (5.27)$$

From (2.9), $\mathbb{P}$-a.s., $\forall (b,b_0) \in \mathbb{R}^2$, $\forall \xi \in \Gamma_K$, it is bounded and there exists a polynomial $p(\cdot)$ independent of $(b_0,b)$ s.t.:

$$\|\mathbf{W}(b,b_0,\omega,\xi)\| \leq |p(\xi)|.$$

**Lemma 5.7.** *$\mathbb{P}$-a.s. on $\Omega$, $\forall (b_0,b) \in \mathbb{R}^2$, $\forall \xi \in \Gamma_K$, $\mathbf{P}(b)\cdot\mathbf{W}(b,b_0,\omega,\xi)$ is bounded and there exists a polynomial $p(\cdot)$ s.t. $\forall b \in \mathbb{R}^2$, $|b-b_0|$ small enough and $\forall \xi \in \Gamma_K$:*

$$\|\mathbf{P}(b)\cdot\mathbf{W}(b,b_0,\omega,\xi)\| \leq |p(\xi)|. \quad (5.28)$$

*Proof.* Similarly to the proof of Lemma 3.2, then $\mathbb{P}$-a.s. , $\forall (b,b_0) \in \mathbb{R}^2$ and $\forall \xi \in \Gamma_K$, we have in the bounded operators sense (see also [11, Prop. 3.2]):

$$R_\infty(b,\omega,\xi) = \tilde{R}_\infty(b,b_0,\omega,\xi) - R_\infty(b,\omega,\xi)\tilde{T}_\infty(b,b_0,\omega,\xi), \quad (5.29)$$

where $\tilde{R}_\infty(b,b_0,\omega,\xi)$ is the operator generated by the kernel defined in (3.9) with $L = \infty$ and:

$$\tilde{T}_\infty(b,b_0,\omega,\xi) := \delta b\tilde{T}_{1,\infty}(b,b_0,\omega,\xi) + (\delta b)^2\tilde{T}_{2,\infty}(b,b_0,\omega,\xi), \quad (5.30)$$

with $\tilde{T}_{j,\infty}(b, b_0, \omega, \xi)$ the operator generated by the kernel defined in (3.10) with $L = \infty$. Notice that due to (2.9), $\mathbb{P}$-a.s., there exists a polynomial $p(\cdot)$ s.t. $\forall (b, b_0) \in \mathbb{R}^2$, $\forall \xi \in \Gamma_K$:

$$\|\tilde{R}_\infty(b, b_0, \xi)\|, \|\tilde{T}_{2,\infty}(b, b_0, \xi)\| \leq |p(\xi)|. \tag{5.31}$$

Now we remove the $(\omega, \xi)$-dependence in the notations. From (5.29), we have on $\mathbb{R}^6 \setminus D_\infty$:

$$(\mathbf{P}(b) \cdot \mathbf{W}(b, b_0))(\mathbf{x}, \mathbf{y}) = \mathbf{a}(\mathbf{x} - \mathbf{y}) \cdot \mathbf{P}(b) R_\infty^{(1)}(\mathbf{x}, \mathbf{y}; b) + \mathbf{a}(\mathbf{x} - \mathbf{y}) \cdot \mathbf{P}(b)(R_\infty(b)\tilde{T}_\infty(b, b_0))(\mathbf{x}, \mathbf{y}).$$

The first term in the r.h.s of this expression is the integral kernel of $T_{1,\infty}(b)$ which is bounded, and its norm satisfies (5.7). We want to study the second term. On $\mathbb{R}^6 \setminus D_\infty$, we have:

$$\begin{aligned}\mathbf{a}(\mathbf{x} - \mathbf{y}) \cdot \mathbf{P}(b)(R_\infty(b)\tilde{T}_\infty(b, b_0))(\mathbf{x}, \mathbf{y}) = \int_{\mathbb{R}^3} \mathrm{d}\mathbf{z}\, \mathbf{a}(\mathbf{x} - \mathbf{z}) \cdot \mathbf{P}(b) R_\infty^{(1)}(\mathbf{x}, \mathbf{z}; b)\tilde{T}_\infty(\mathbf{z}, \mathbf{y}; b, b_0) + \\ + \int_{\mathbb{R}^3} \mathrm{d}\mathbf{z}\, \mathbf{P}(b) R_\infty^{(1)}(\mathbf{x}, \mathbf{z}; b)\mathbf{a}(\mathbf{z} - \mathbf{y})\tilde{T}_\infty(\mathbf{z}, \mathbf{y}; b, b_0),\end{aligned} \tag{5.32}$$

The first term of the r.h.s. of (5.32) is the integral kernel of $T_{1,\infty}(b)\tilde{T}_\infty(b, b_0) = \frac{i}{2}(P_1(b)R_\infty(b)P_2(b) - P_2(b)R_\infty(b)P_1(b))R_\infty(b)\tilde{T}_\infty(b, b_0)$ which is bounded due to (5.1) and (5.31) knowing (5.29), its norm is bounded above by a $b$-independent quantity. Furthermore in view of (3.17), on $\mathbb{R}^6 \setminus D_\infty$:

$$\begin{aligned}\mathbf{a}(\mathbf{z} - \mathbf{y})\tilde{T}_\infty(\mathbf{z}, \mathbf{y}; b, b_0) &= \mathrm{e}^{i\delta b \phi(\mathbf{z}, \mathbf{y})}\mathbf{a}(\mathbf{z} - \mathbf{y})\Big(\delta b \mathbf{P}(b) \cdot \mathbf{a}(\mathbf{z} - \mathbf{y}) + \frac{(\delta b)^2}{2}\mathbf{a}^2(\mathbf{z} - \mathbf{y})\Big) R_\infty^{(1)}(\mathbf{z}, \mathbf{y}; b_0) \\ &= \delta b \mathbf{P}(b)\mathrm{e}^{i\delta b \phi(\mathbf{z}, \mathbf{y})}\mathbf{a}^2(\mathbf{z} - \mathbf{y})R_\infty^{(1)}(\mathbf{z}, \mathbf{y}; b_0) - \frac{1}{2}(\delta b)^2 \mathrm{e}^{i\delta b \phi(\mathbf{z}, \mathbf{y})}\mathbf{a}^3(\mathbf{z} - \mathbf{y})R_\infty^{(1)}(\mathbf{z}, \mathbf{y}; b_0),\end{aligned} \tag{5.33}$$

which is the integral kernel of $2\delta b \mathbf{P}(b)\tilde{T}_{2,\infty}(b, b_0) - (\delta b)^2 \mathbf{Y}(b, b_0)$, where $\mathbf{Y}(b, b_0) = \mathbf{Y}(b, b_0, \omega, \xi)$ is the operator defined via its integral kernel on $\mathbb{R}^6 \setminus D_\infty$:

$$\mathbf{Y}(\mathbf{x}, \mathbf{y}; b, b_0, \omega, \xi) := \frac{1}{2}\mathbf{a}^2(\mathbf{x} - \mathbf{y})\mathbf{W}(\mathbf{x}, \mathbf{y}; b, b_0, \omega, \xi). \tag{5.34}$$

Notice that from (2.9) and (5.27), $\|\mathbf{Y}(b, b_0, \omega, \xi)\| \leq |p(\xi)|$ for some polynomial $p(\cdot)$ independent of $(b, b_0)$. Since $\mathbf{Y}(b, b_0)$ is bounded, then by Lemmas 5.2 and 5.1 the second term of the r.h.s. of (5.32) is the kernel of the bounded operator $\delta b \mathbf{P}(b) R_\infty(b)\big(2\mathbf{P}(b)\tilde{T}_{2,\infty}(b, b_0) - \delta b \mathbf{Y}(b, b_0)\big)$ and its norm is bounded above by a $b$-independent polynomial in $\xi$. ∎

*Proof of Proposition 5.6.* Let $j = 0$. Define $\tilde{\mathcal{I}}_0(b, b_0, \omega, \xi)$ as in (5.14) but we replace each operator $R_\infty(b, .)$ with $\tilde{R}_\infty(b, b_0, .)$. Then from (5.29) and (5.14), $\mathbb{P}$-a.s., $\forall (b, b_0) \in \mathbb{R}^2$, $\forall \xi \in \Gamma_K$:

$$\mathcal{I}_0(b, \omega, \xi) - \mathcal{I}_0(b_0, \omega, \xi) = \tilde{\mathcal{I}}_0(b, b_0, \omega, \xi) - \mathcal{I}_0(b_0, \omega, \xi) - \tilde{\mathcal{R}}_0(b, b_0, \omega, \xi),$$

where $\tilde{\mathcal{R}}_0(b, b_0, \omega, \xi)$ is the following bounded operator:

$$\begin{aligned}(\tilde{R}_\infty(\cdot, \xi_0))^2 R_\infty(\cdot, \xi)\tilde{T}_\infty(\cdot, \xi) + R_\infty(\cdot, \xi_0)\tilde{T}_\infty(\cdot, \xi_0)R_\infty(\cdot, \xi_0)R_\infty(\cdot, \xi) + \\ + \tilde{R}_\infty(\cdot, \xi_0)R_\infty(\cdot, \xi_0)\tilde{T}_\infty(\cdot, \xi_0)R_\infty(\cdot, \xi).\end{aligned} \tag{5.35}$$

Firstly in the kernel sense, for any $\mathbf{x} \in \mathbb{R}^3$:

$$\begin{aligned}\tilde{\mathcal{I}}_0(\mathbf{x}, \mathbf{x}; b, b_0, \omega, \xi) - \mathcal{I}_0(\mathbf{x}, \mathbf{x}; b_0, \omega, \xi) = \int_{\mathbb{R}^6} \mathrm{d}\mathbf{z}_1 \mathrm{d}\mathbf{z}_2\, \{\mathrm{e}^{i\delta b(\phi(\mathbf{x}, \mathbf{z}_1) + \phi(\mathbf{z}_1, \mathbf{z}_2) + \phi(\mathbf{z}_2, \mathbf{x}))} - 1\} \times \\ \times R_\infty^{(1)}(\mathbf{x}, \mathbf{z}_1; b_0, \omega, \xi_0) R_\infty^{(1)}(\mathbf{z}_1, \mathbf{z}_2; b_0, \omega, \xi) R_\infty^{(1)}(\mathbf{z}_2, \mathbf{x}; b_0, \omega, \xi_0).\end{aligned}$$

Since $|e^{i\delta b(\phi(\mathbf{x},\mathbf{z}_1)+\phi(\mathbf{z}_1,\mathbf{z}_2)+\phi(\mathbf{z}_2,\mathbf{x}))}-1| \leq |\delta b||\mathbf{x}-\mathbf{z}_1||\mathbf{z}_1-\mathbf{z}_2|$, then (2.9) implies that $\mathbb{P}$-a.s., there exists a polynomial $p(\cdot)$ s.t. $\forall \xi \in \Gamma_K$, $\forall \mathbf{x} \in \mathbb{R}^3$, $\forall b \in \mathbb{R}$ with $|b-b_0|$ small enough:

$$|\tilde{\mathcal{I}}_0(\mathbf{x},\mathbf{x};b,b_0,\omega,\xi) - \mathcal{I}_0(\mathbf{x},\mathbf{x};b_0,\omega,\xi)| \leq |\delta b||p(\xi)|. \tag{5.36}$$

Let us now estimate $\mathrm{Tr}_{L^2(\mathbb{R}^3)}\{\chi_{\Lambda_L}\tilde{\mathcal{R}}_0(b,b_0,\omega,\xi)\chi_{\Lambda_L}\}$. Consider the following term in (5.35):

$$r_0(b,b_0,\omega,\xi) := \left(\tilde{R}_\infty(b,b_0,\omega,\xi_0)\right)^2 R_\infty(b,\omega,\xi)\tilde{T}_\infty(b,b_0,\omega,\xi).$$

By using (3.10), (3.11) with $L=\infty$ and (3.17), we have on $\mathbb{R}^6$:

$$\begin{aligned}(R_\infty(b,\omega,\xi)\tilde{T}_\infty(b,b_0,\omega,\xi))(\mathbf{x},\mathbf{y}) &= -(\delta b)^2 (R_\infty(b,\omega,\xi)\tilde{T}_{2,\infty}(b,b_0,\omega,\xi))(\mathbf{x},\mathbf{y}) + \\ &+ \delta b \int_{\mathbb{R}^3} \mathrm{d}\mathbf{z}\, R_\infty^{(1)}(\mathbf{x},\mathbf{z};b,\omega,\xi)(i\nabla_{\mathbf{z}} + b\mathbf{a}(\mathbf{z}))\cdot \mathbf{a}(\mathbf{z}-\mathbf{y}) e^{i\delta b\phi(\mathbf{z},\mathbf{y})} R_\infty^{(1)}(\mathbf{z},\mathbf{y};b_0,\omega,\xi).\end{aligned}$$

Then

$$R_\infty(b,\omega,\xi)\tilde{T}_\infty(b,b_0,\omega,\xi) = \delta b R_\infty(b,\omega,\xi)\Big(\mathbf{P}(b)\cdot \mathbf{W}(b,b_0,\omega,\xi) - \delta b \tilde{T}_{2,\infty}(b,b_0,\omega,\xi)\Big), \tag{5.37}$$

where $\mathbf{W}(b,b_0,\omega,\xi)$ is defined in (5.27). Hence we have to estimate:

$$\left\|\chi_{\Lambda_L}\left(\tilde{R}_\infty(b,b_0,\omega,\xi_0)\right)^2 R_\infty(b,\omega,\xi)\left(\mathbf{P}(b)\cdot\mathbf{W}(b,b_0,\omega,\xi)\chi_{\Lambda_L} - \tilde{T}_{2,\infty}(b,b_0,\omega,\xi)\chi_{\Lambda_L}\right)\right\|_{\mathfrak{I}_1}.$$

In view of (5.27) and (2.9), $\mathbb{P}$-a.s., there exists a polynomial $p(\cdot)$ s.t. $\forall \xi \in \Gamma_K$, $\forall b \in \mathbb{R}$ and $\forall L \in (0,\infty)$:

$$\|\chi_{\Lambda_L}\tilde{R}_\infty(b,b_0,\omega,\xi_0)\|_{\mathfrak{I}_2},\ \|\mathbf{W}(b,b_0,\omega,\xi)\chi_{\Lambda_L}\|_{\mathfrak{I}_2},\ \|\tilde{T}_{2,\infty}(b,b_0,\omega,\xi)\chi_{\Lambda_L}\|_{\mathfrak{I}_2} \leq |p(\xi)|L^{\frac{3}{2}}. \tag{5.38}$$

Then from (5.4) and (5.31), $\mathbb{P}$-a.s., there exists another polynomial $p(\cdot)$ s.t. $\forall \xi \in \Gamma_K$, $\forall b \in \mathbb{R}$ with $|b-b_0|$ small enough and $\forall L \in (0,\infty)$:

$$|\Lambda_L|^{-1}|\mathrm{Tr}_{L^2(\mathbb{R}^3)}\{\chi_{\Lambda_L} r_0(b,b_0,\omega,\xi)\chi_{\Lambda_L}\}| \leq |\delta b||p(\xi)|. \tag{5.39}$$

Now consider the operator $r_1(b,b_0,\omega,\xi) := R_\infty(\cdot,\xi_0)\tilde{T}_\infty(\cdot,\xi_0)R_\infty(\cdot,\xi_0)R_\infty(\cdot,\xi)$. From (5.37):

$$r_1(b,b_0,\omega,\xi) = \delta b R_\infty(\cdot,\xi_0)\Big(\mathbf{P}(\cdot)\cdot\mathbf{W}(\cdot,\xi_0) - \delta b\tilde{T}_{2,\infty}(\cdot,\xi_0)\Big)R_\infty(\cdot,\xi_0)R_\infty(\cdot,\xi).$$

Then by using (5.28) and the above arguments, we conclude that $\mathbb{P}$-a.s., there exists a polynomial $p(\cdot)$ s.t. $\forall \xi \in \Gamma_K$, $\forall b \in \mathbb{R}$ with $|b-b_0|$ small enough and $\forall L \in (0,\infty)$:

$$|\Lambda_L|^{-1}|\mathrm{Tr}_{L^2(\mathbb{R}^3)}\{\chi_{\Lambda_L} r_1(b,b_0,\omega,\xi)\chi_{\Lambda_L}\}| \leq |\delta b||p(\xi)|. \tag{5.40}$$

This also holds for the last term of (5.35) and then $|\Lambda_L|^{-1}|\mathrm{Tr}_{L^2(\mathbb{R}^3)}\{\chi_{\Lambda_L}\tilde{\mathcal{R}}_0(b,b_0,\omega,\xi)\chi_{\Lambda_L}\}| \leq |\delta b||p(\xi)|$. This together with (5.36) prove *i)* with $j=0$. Let us show *ii)*. From (5.17)-(5.18), (5.30), (5.35) and Lemma 7.2 *ii)*, then $\mathbb{P}$-a.s., there exists a polynomial $p(\cdot)$ s.t. $\forall \xi \in \Gamma_K$, $\forall b \in \mathbb{R}$ with $|b-b_0|$ small enough:

$$\forall \mathbf{x} \in \mathbb{R}^3, \quad |\tilde{\mathcal{R}}_0(\mathbf{x},\mathbf{x};b,b_0,\omega,\xi)| \leq |\delta b||p(\xi)|(1+|\mathbf{x}|^\alpha).$$

This together with (5.36) imply:

$$|\mathcal{I}_0(\mathbf{0},\mathbf{0};b,\omega,\xi) - \mathcal{I}_0(\mathbf{0},\mathbf{0};b_0,\omega,\xi)| \leq |\delta b||p(\xi)|. \tag{5.41}$$

Then *ii)* with $j=0$ follows from (5.41) and the $\omega$-independent estimate (5.15).

Let $j=1$. Define the function on $\mathbb{R}^3$:

$$\tilde{\mathcal{I}}_1(\mathbf{x},\mathbf{x};b,b_0,\omega,\xi) := \int_{\mathbb{R}^3} \mathrm{d}\mathbf{z}\, \tilde{R}_\infty^{(1)}(\mathbf{x},\mathbf{z};b,b_0,\omega,\xi)\mathbf{a}(\mathbf{z}-\mathbf{x})\cdot(i\nabla_{\mathbf{z}} + b\mathbf{a}(\mathbf{z}))\tilde{R}_\infty^{(1)}(\mathbf{z},\mathbf{x};b,b_0,\omega,\xi).$$

Due to (5.17)-(5.18) and (3.1), $\mathbb{P}$-a.s., $\forall (b, b_0) \in \mathbb{R}^2$, $\forall \xi \in \Gamma_K$, it is well-defined since by (3.17):

$$\tilde{\mathcal{I}}_1(\mathbf{x},\mathbf{x};b,b_0,\omega,\xi) = \int_{\mathbb{R}^3} \mathrm{d}\mathbf{z}\, R_\infty^{(1)}(\mathbf{x},\mathbf{z};b_0,\omega,\xi) T_{1,\infty}(\mathbf{z},\mathbf{x};b_0,\omega,\xi) + \\ + 2\delta b \int_{\mathbb{R}^3} \mathrm{d}\mathbf{z}\, R_\infty^{(1)}(\mathbf{x},\mathbf{z};b_0,\omega,\xi) T_{2,\infty}(\mathbf{z},\mathbf{x};b_0,\omega,\xi). \quad (5.42)$$

From the definition (5.12) of $\mathcal{I}_1$, we replace the resolvent on the left and the one in $T_{1,\infty}$ (see (1.12)) with the r.h.s. of (5.29). Then $\forall \mathbf{x} \in \mathbb{R}^3$:

$$\mathcal{I}_1(\mathbf{x},\mathbf{x};b,\omega,\xi) - \mathcal{I}_1(\mathbf{x},\mathbf{x};b_0,\omega,\xi) = \tilde{\mathcal{I}}_1(\mathbf{x},\mathbf{x};b,b_0,\omega,\xi) - \mathcal{I}_1(\mathbf{x},\mathbf{x};b_0,\omega,\xi) + \tilde{\mathcal{R}}_1(\mathbf{x},\mathbf{x};b,b_0,\omega,\xi),$$

where $\tilde{\mathcal{R}}_1(\cdot,\cdot;b,b_0,\omega,\xi) := s_0(\cdot,\cdot;b,b_0,\omega,\xi) + s_1(\cdot,\cdot;b,b_0,\omega,\xi)$ and

$$s_0(\cdot,\cdot;b,b_0,\omega,\xi) := \big(R_\infty(b,\omega,\xi)\tilde{T}_\infty(b,b_0,\omega,\xi)T_{1,\infty}(b,\omega,\xi)\big)(\cdot,\cdot), \quad (5.43)$$

$$s_1(\cdot,\cdot;b,b_0,\omega,\xi) := \int_{\mathbb{R}^3} \mathrm{d}\mathbf{z}\, \tilde{R}_\infty^{(1)}(\cdot,\mathbf{z};b,b_0,\omega,\xi)\mathbf{a}(\mathbf{z}-\cdot)\cdot\mathbf{P}(b)\big(R_\infty(b,\omega,\xi)\tilde{T}_\infty(b,b_0,\omega,\xi)\big)(\mathbf{z},\cdot). \quad (5.44)$$

First by (5.42), we have for any $\mathbf{x} \in \mathbb{R}^3$:

$$\tilde{\mathcal{I}}_1(\mathbf{x},\mathbf{x};b,b_0,\omega,\xi) - \mathcal{I}_1(\mathbf{x},\mathbf{x};b_0,\omega,\xi) = 2\delta b\big((R_\infty(b_0,\omega,\xi)T_{2,\infty}(b_0,\omega,\xi)\big)(\mathbf{x},\mathbf{x}).$$

Then by (5.17)-(5.18), $\mathbb{P}$-a.s., there exists a polynomial $p(\cdot\,)$ s.t. $\forall \xi \in \Gamma_K$, $\forall b \in \mathbb{R}$, $\forall \mathbf{x} \in \mathbb{R}^3$:

$$|\tilde{\mathcal{I}}_1(\mathbf{x},\mathbf{x};b,b_0,\omega,\xi) - \mathcal{I}_1(\mathbf{x},\mathbf{x};b_0,\omega,\xi)| \leq |\delta b||p(\xi)|. \quad (5.45)$$

From (5.37), (5.43) is the diagonal part of the integral kernel of the bounded operator:

$$s_0(b,b_0,\omega,\xi) := \frac{i\delta b}{2} R_\infty(b,\omega,\xi)\Big(\mathbf{P}(b)\cdot\mathbf{W}(b,b_0,\omega,\xi) - \delta b\tilde{T}_{2,\infty}(b,b_0,\omega,\xi)\Big)\times \\ \Big(P_1(b)R_\infty(b,\omega,\xi)P_2(b) - P_2(b)R_\infty(b,\omega,\xi)P_1(b)\Big)R_\infty(b,\omega,\xi).$$

By using again the same arguments as above together with (5.1), (5.28) and (5.31), then $\mathbb{P}$-a.s. there exists a polynomial $p(\cdot\,)$ s.t. $\forall \xi \in \Gamma_K$, $\forall b \in \mathbb{R}$ with $|b-b_0|$ small enough and $\forall L \in (0,\infty)$:

$$|\Lambda_L|^{-1}|\mathrm{Tr}_{L^2(\mathbb{R}^3)}\{\chi_{\Lambda_L} s_0(b,b_0,\omega,\xi)\chi_{\Lambda_L}\}| \leq |\delta b||p(\xi)|. \quad (5.46)$$

Similary to the proof of Lemma 5.7. Then (3.17), (5.9), (5.33) and (5.37), show that (5.44) is the diagonal part of the integral kernel of the bounded operator :

$$s_1(b,b_0,\omega,\xi) = \frac{i}{2}\tilde{R}_\infty(b,b_0,\xi)\Big(P_1(b)R_\infty(b,\xi)P_2(b) - P_2(b)R_\infty(b,\xi)P_1(b)\Big)R_\infty(b,\xi)\tilde{T}_\infty(b,b_0,\xi) + \\ + \delta b\tilde{R}_\infty(b,b_0,\xi)\mathbf{P}(b)R_\infty(b,\xi)\Big(2\mathbf{P}(b)\tilde{T}_{2,\infty}(b,b_0,\xi) - \delta b\mathbf{Y}(b,b_0,\xi)\Big),$$

with $\mathbf{Y}(b,b_0,\xi)$ the operator defined in (5.34). Due to (5.1), the first term in the above r.h.s. can be treated exactly as the operator $r_0(b,b_0,\omega,\xi)$ at the beginning of this proof. For the second term, we have (5.38) and under the same conditions, $\|\mathbf{Y}(b,b_0,\xi)\chi_{\Lambda_L}\|_{\mathfrak{I}_2} \leq |p(\xi)|L^{\frac{3}{2}}$. Therefore $\mathbb{P}$-a.s., there exists another polynomial $p(\cdot\,)$ s.t. $\forall \xi \in \Gamma_K$, $\forall b \in \mathbb{R}$ with $|b-b_0|$ small enough and $\forall L \in (0,\infty)$:

$$|\Lambda_L|^{-1}|\mathrm{Tr}_{L^2(\mathbb{R}^3)}\{\chi_{\Lambda_L} s_1(b,b_0,\omega,\xi)\chi_{\Lambda_L}\}| \leq |\delta b||p(\xi)|.$$

Due to (5.46), this also holds for $|\Lambda_L|^{-1}|\mathrm{Tr}_{L^2(\mathbb{R}^3)}\{\chi_{\Lambda_L}\tilde{\mathcal{R}}_1(b,b_0,\omega,\xi)\chi_{\Lambda_L}\}|$. This together with (5.45) prove $i)$ with $j=1$. Now consider (5.43)-(5.44). From (5.17)-(5.18) and Lemma 7.2 $ii)$, $\mathbb{P}$-a.s., there exists a polynomial $p(\cdot\,)$ s.t. $\forall \xi \in \Gamma_K$, $\forall b \in \mathbb{R}$, $|b-b_0|$ small enough:

$$\forall \mathbf{x} \in \mathbb{R}^3, \quad |\tilde{\mathcal{R}}_1(\mathbf{x},\mathbf{x};b,b_0,\omega,\xi)| \leq |\delta b||p(\xi)|(1+|\mathbf{x}|^\alpha)^2.$$

This together with (5.45) imply:

$$|\mathcal{I}_1(\mathbf{0},\mathbf{0};b,\omega,\xi)-\mathcal{I}_1(\mathbf{0},\mathbf{0};b_0,\omega,\xi)| \leq |\delta b||p(\xi)|. \tag{5.47}$$

Then $ii)$ follows from (5.47) and the $\omega$-independent estimate (5.15). ■

## 6. Proof of Theorem 1.1 and Theorem 1.2

In the whole of this section, for any compact subset $K$ of $\mathcal{D}_\epsilon$ and any $\beta \in [\beta_1,\beta_2] \in \mathbb{R}$, $\Gamma_K$ is defined for $\beta=\beta_2$. We will use $\eta := \min\{E_0-E_K, \frac{\vartheta_K}{2\beta_2}\}$. We first give two important results proven subsequently.

**Proposition 6.1.** *i) $\mathbb{P}$-a.s. on $\Omega$, $\forall b \in \mathbb{R}$, $\forall 0<\beta_1<\beta_2$, $\forall K$ compact subset of $\mathcal{D}_\epsilon$, one has:*

$$\lim_{L\to\infty}\frac{1}{|\Lambda_L|}\int_{\Lambda_L}\mathrm{d}\mathbf{x}\,\mathcal{L}^{(\omega)}_{\infty,n}(\mathbf{x},\mathbf{x};\beta,b,z,\epsilon) = \mathbb{E}\big[\mathcal{L}^{(\omega)}_{\infty,n}(\mathbf{0},\mathbf{0};\beta,b,z,\epsilon)\big] \quad n=0,1,$$

*uniformly in $(\beta,z)\in[\beta_1,\beta_2]\times K$.*
*ii) $\mathbb{P}$-a.s. on $\Omega$, $\forall 0<\beta_1<\beta_2$, $\forall K$ compact subset of $\mathcal{D}_\epsilon$, one has:*

$$\lim_{L\to\infty}\frac{1}{|\Lambda_L|}\int_{\Lambda_L}\mathrm{d}\mathbf{x}\,\mathcal{L}^{(\omega)}_{\infty,2}(\mathbf{x},\mathbf{x};\beta,0,z,\epsilon) = \mathbb{E}\big[\mathcal{L}^{(\omega)}_{\infty,2}(\mathbf{0},\mathbf{0};\beta,0,z,\epsilon)\big],$$

*uniformly in $(\beta,z)\in[\beta_1,\beta_2]\times K$.*

**Proposition 6.2.** *$\mathbb{P}$-a.s. on $\Omega$, $\forall b \in \mathbb{R}$, $\forall 0<\beta_1<\beta_2$, $\forall K$ compact subset of $\mathcal{D}_\epsilon$, one has:*

$$\lim_{L\to\infty}\frac{1}{|\Lambda_L|}\frac{1}{\beta}\left|\int_{\Lambda_L}\mathrm{d}\mathbf{x}\left(\frac{\partial^n\mathcal{L}^{(\omega)}_L}{\partial b^n}(\mathbf{x},\mathbf{x};\beta,b,z,\epsilon)-\mathcal{L}^{(\omega)}_{\infty,n}(\mathbf{x},\mathbf{x};\beta,b,z,\epsilon)\right)\right| = 0 \quad n=0,1,2, \tag{6.1}$$

*uniformly in $(\beta,z)\in[\beta_1,\beta_2]\times K$.*

### 6.1. **Proof of Theorem 1.1.**

6.1.1. *Proof of $i)$.* The results follow from Propositions 6.1 and 6.2, with the simple relation:

$$\mathcal{X}^{(\omega)}_{L,n}(\beta,b,z,\epsilon) = \left(\frac{q}{c}\right)^n\frac{\epsilon}{\beta|\Lambda_L|}\int_{\Lambda_L}\mathrm{d}\mathbf{x}\left(\frac{\partial^n\mathcal{L}^{(\omega)}_L}{\partial b^n}(\mathbf{x},\mathbf{x};\beta,b,z,\epsilon)-\mathcal{L}^{(\omega)}_{\infty,n}(\mathbf{x},\mathbf{x};\beta,b,z,\epsilon)\right)+ \\ +\left(\frac{q}{c}\right)^n\frac{\epsilon}{\beta|\Lambda_L|}\int_{\Lambda_L}\mathrm{d}\mathbf{x}\,\mathcal{L}^{(\omega)}_{\infty,n}(\mathbf{x},\mathbf{x};\beta,b,z,\epsilon). \tag{6.2}$$

6.1.2. *Proof of ii).* Define the operator on $L^2(\mathbb{R}^3)$:

$$\mathcal{J}_\infty^{(\omega)}(\beta,b,z,\epsilon) := \frac{i}{2\pi}\int_{\Gamma_K} \mathrm{d}\xi\, \frac{z\mathrm{e}^{-\beta\xi}}{1+\epsilon z\mathrm{e}^{-\beta\xi}} R_\infty(b,\omega,\xi).$$

Since the function $\xi \mapsto \frac{z\mathrm{e}^{-\beta\xi}}{1+\epsilon z\mathrm{e}^{-\beta\xi}}$ is also exponentially decreasing on $\Gamma_K$ when $|\Re\xi| \to \infty$, the study of the operator $\mathcal{J}_\infty^{(\omega)}(\beta,b,z,\epsilon)$ is therefore similar to the one of $\mathcal{L}_{\infty,0}^{(\omega)}(\beta,b,z,\epsilon)$. Then we deduce that $\mathbb{P}$-a.s., $\forall 0<\beta_1<\beta_2$, $\forall b\in\mathbb{R}$ and for any compact subset $K$ of $\mathcal{D}_\epsilon$:

$$\rho_\infty(\beta,b,z,\epsilon) := \lim_{L\to\infty} \rho_L^{(\omega)}(\beta,b,z,\epsilon) = \mathbb{E}[\mathcal{J}_\infty^{(\omega)}(\mathbf{0},\mathbf{0};\beta,b,z,\epsilon)], \tag{6.3}$$

uniformly in $(\beta,z)\in[\beta_1,\beta_2]\times K$. Here $\mathcal{J}_\infty^{(\omega)}(\cdot\,,\cdot\,;\beta,b,z,\epsilon)$ is the integral kernel of $\mathcal{J}_\infty^{(\omega)}(\beta,b,z,\epsilon)$.

Let $K'$ be the interior of $K$ which is supposed nonempty. Fix $z_0\in K'$. Let $z\in K$ near $z_0$. Since $\mathbb{P}$-a.s., $\forall\beta>0$ and $\forall b\in\mathbb{R}$, $P_L^{(\omega)}(\beta,b,\cdot\,,\epsilon)$ is an analytic function on $\mathcal{D}_\epsilon$, then:

$$P_L^{(\omega)}(\beta,b,z,\epsilon) = P_L^{(\omega)}(\beta,b,z_0,\epsilon) + (z-z_0)\frac{1}{\beta z_0}\rho_L^{(\omega)}(\beta,b,z_0,\epsilon) + o((z-z_0)). \tag{6.4}$$

Notice that since $\|R_L(b,\omega,\xi)\| \le c, \forall\xi\in\Gamma_K$ and $L\in(0,\infty]$, the third term in the r.h.s. of (6.4) is $L$-independent. It follows from Theorem 1.1 *i)*, $\mathbb{P}$-a.s., $\forall\beta>0$, $\forall b\in\mathbb{R}$ and for $z\in K$ near $z_0$, the following identity between non-random limits:

$$P_\infty(\beta,b,z,\epsilon) - P_\infty(\beta,b,z_0,\epsilon) = (z-z_0)\frac{1}{\beta z_0}\mathbb{E}[\mathcal{J}_\infty^{(\omega)}(\mathbf{0},\mathbf{0};\beta,b,z_0,\epsilon)] + o(z-z_0).$$

So $P_\infty(\beta,b,\cdot\,,\epsilon)$ is analytic in $z_0$, and $\beta z_0\frac{\partial P_\infty}{\partial z}(\beta,b,z_0,\epsilon) = \mathbb{E}[\mathcal{J}_\infty^{(\omega)}(\mathbf{0},\mathbf{0};\beta,b,z_0,\epsilon)]$.

6.1.3. *Proof of iii).* Let $\alpha\in(0,\frac{1}{3})$ and $b_0\in\mathbb{R}$ be fixed. Since $\mathbb{P}$-a.s., $\forall\beta>0$ and $\forall z\in\mathcal{D}_\epsilon$, $b\mapsto P_L^{(\omega)}(\beta,b,z,\epsilon)$ is a $\mathcal{C}^\infty$-function, then for $b\in\mathbb{R}$ near $b_0$:

$$P_L^{(\omega)}(\beta,b,z,\epsilon) = P_L^{(\omega)}(\beta,b_0,z,\epsilon) + (b-b_0)\frac{c}{q}\mathcal{X}_{L,1}^{(\omega)}(\beta,b_0,z,\epsilon) + o((b-b_0)).$$

In virtue of Remark 6.5 below, the third term in the r.h.s. of the above equality is uniformly bounded in $L\in(0,\infty)$. It follows from Theorem 1.1 *i)*, $\mathbb{P}$-a.s., $\forall\beta>0$, $\forall z\in\mathcal{D}_\epsilon$ and for $b\in\mathbb{R}$ sufficiently close to $b_0$, the following relation:

$$P_\infty(\beta,b,z,\epsilon) - P_\infty(\beta,b_0,z,\epsilon) = (b-b_0)\frac{c}{q}\mathcal{X}_{\infty,1}(\beta,b_0,z,\epsilon) + o((b-b_0)).$$

Then $P_\infty(\beta,\cdot\,,z,\epsilon)$ is differentiable at $b_0$ and $\frac{\partial P_\infty}{\partial b}(\beta,b_0,z,\epsilon) = (\frac{c}{q})\mathcal{X}_{\infty,1}(\beta,b_0,z,\epsilon)$.

6.2. **Proof of Theorem 1.2.** *i)* follows from (6.2) for $n=2$ and $b=0$, Propositions 6.1 *ii)* and 6.2. Let us prove *ii)*. Let $\alpha\in(0,\frac{1}{4})$. Since $\mathbb{P}$-a.s. $\forall\beta>0$ and $\forall z\in\mathcal{D}_\epsilon$, $b\mapsto\mathcal{X}_{L,1}^{(\omega)}(\beta,b,z,\epsilon)$ is a $\mathcal{C}^\infty$-function near $b_0=0$, then for real $b$ sufficiently small:

$$\mathcal{X}_{L,1}^{(\omega)}(\beta,b,z,\epsilon) = \mathcal{X}_{L,1}^{(\omega)}(\beta,0,z,\epsilon) + b\frac{c}{q}\mathcal{X}_{L,2}^{(\omega)}(\beta,0,z,\epsilon) + o(b).$$

From Remarks 4.4 and 6.5, the last term in the r.h.s. of this equality is uniformly bounded in $L\in(0,\infty)$. It follows from Theorems 1.1 *i)* and 1.2 *i)*:

$$\mathcal{X}_{\infty,1}(\beta,b,z,\epsilon) - \mathcal{X}_{\infty,1}(\beta,0,z,\epsilon) = b\frac{c}{q}\mathcal{X}_{\infty,2}(\beta,0,z,\epsilon) + o(b).$$

This together with Theorem 1.1 *iii)*, imply that $P_\infty(\beta,\cdot\,,z,\epsilon)$ is twice differentiable near $b=0$ and $\frac{\partial^2 P_\infty}{\partial b^2}(\beta,0,z,\epsilon) = (\frac{c}{q})^2\mathcal{X}_{\infty,2}(\beta,0,z,\epsilon)$.

6.3. **Proof of Proposition 6.1.** By Corollary 5.4, we can apply the Birkhoff-Khintchine theorem [21, Prop. 1.13] to $(\omega, \mathbf{x}) \mapsto \mathcal{L}^{(\omega)}_{\infty,n}(\mathbf{x}, \mathbf{x}; \beta, b, z, \epsilon)$, $n = 0, 1, 2$. This implies that $\mathbb{P}$-a.s., for any $\beta > 0$ and $z \in K$:

$$\lim_{L\to\infty} \frac{1}{|\Lambda_L|} \int_{\Lambda_L} \mathrm{d}\mathbf{x}\, \mathcal{L}^{(\omega)}_{\infty,n}(\mathbf{x}, \mathbf{x}; \beta, b, z, \epsilon) = \mathbb{E}\big[\mathcal{L}^{(\omega)}_{\infty,n}(\mathbf{0}, \mathbf{0}; \beta, b, z, \epsilon)\big] \quad n = 0, 1, 2.$$

But this is not sufficient to prove Proposition 6.1. Consider the case of $n = 0$. In view of (5.23) and Proposition 5.3, we can use the Birkhoff-Khintchine theorem for $(\mathbf{x}, \omega) \mapsto \mathcal{I}_0(\mathbf{x}, \mathbf{x}; b, \omega, \xi)$. Then $\forall b \in \mathbb{R}$ and $\forall \xi \in \Gamma_K$, there exists $\Omega_{\xi,b} \subset \Omega$ with $\mathbb{P}(\Omega_{\xi,b}) = 1$ s.t. $\forall \omega \in \Omega_{\xi,b}$:

$$\lim_{L\to\infty} \frac{1}{|\Lambda_L|} \int_{\Lambda_L} \mathrm{d}\mathbf{x}\, \mathcal{I}_0(\mathbf{x}, \mathbf{x}; b, \omega, \xi) = \mathbb{E}\big[\mathcal{I}_0(\mathbf{0}, \mathbf{0}; b, \omega, \xi)\big]. \tag{6.5}$$

Now choose a countable dense subset of $\Gamma_K$, $\Gamma_c := \{\xi_i, i \in \mathbb{N}\}$. Then $\forall b \in \mathbb{R}$, there exists $\Omega_b \subset \Omega$ with $\mathbb{P}(\Omega_b) = 1$ s.t. $\forall \omega \in \Omega_b$ and $\forall \xi_i \in \Gamma_c$, (6.5) holds. Next we use Proposition 5.5 *i*): $\forall b \in \mathbb{R}$, there exists $\Omega'_b \subset \Omega_b$ with $\mathbb{P}(\Omega'_b) = 1$ s.t. $\forall \omega \in \Omega'_b$, the map $\xi \in \Gamma_K \mapsto \varsigma_L(b, \omega, \xi) := \frac{1}{|\Lambda_L|} \int_{\Lambda_L} \mathrm{d}\mathbf{x}\, \mathcal{I}_0(\mathbf{x}, \mathbf{x}; b, \omega, \xi)$ is continuous uniformly in $L \in (0, \infty)$. Let $\xi \in \Gamma_K$. Then $\forall \varepsilon > 0$ there exists $\xi_i \in \Gamma_c$ s.t. $\forall L \in (0, \infty)$, $|\varsigma_L(b, \omega, \xi) - \varsigma_L(b, \omega, \xi_i)| \leq \varepsilon$. On the other hand by Proposition 5.5 *ii*), $\xi_i$ can also be chosen s.t. $|\mathbb{E}\big[\mathcal{I}_0(\mathbf{0}, \mathbf{0}; b, \omega, \xi)\big] - \mathbb{E}\big[\mathcal{I}_0(\mathbf{0}, \mathbf{0}; b, \omega, \xi_i)\big]| \leq \varepsilon$. Then taking the real part we get:

$$\Re\{\mathbb{E}\big[\mathcal{I}_0(\mathbf{0}, \mathbf{0}; b, \omega, \xi)\big]\} - 2\varepsilon \leq \liminf_L \Re\{\varsigma_L(b, \omega, \xi)\} \leq \\ \limsup_L \Re\{\varsigma_L(b, \omega, \xi)\} \leq \Re\{\mathbb{E}\big[\mathcal{I}_0(\mathbf{0}, \mathbf{0}; b, \omega, \xi)\big]\} + 2\varepsilon.$$

Obviously, this also holds true for the imaginary part. Consequently $\forall b \in \mathbb{R}$, $\forall \omega \in \Omega'_b$, (6.5) holds $\forall \xi \in \Gamma_K$. We can repeat the same arguments as above to remove the $b$-dependance but with the use of Proposition 5.6. Then we conclude that $\mathbb{P}$-a.s., $\forall b \in \mathbb{R}$ and $\forall \xi \in \Gamma_K$ (6.5) holds true. Notice that since $\mathbb{P}$-a.s., $\forall b \in \mathbb{R}$ and $\forall \xi \in \Gamma_K$ the integral kernel of $\mathcal{I}_0(b, \omega, \xi)$ is uniformly bounded by a constant independent of $\omega$ and $\xi$, therefore we have:

$$\frac{i}{2\pi} \int_{\Gamma_K} \mathrm{d}\xi\, (\xi - \xi_0)^2 \mathfrak{f}_\epsilon(\beta, z; \xi) \mathbb{E}\big[\mathcal{I}_0(\mathbf{0}, \mathbf{0}; b, \omega, \xi)\big] = \mathbb{E}\big[\mathcal{L}^{(\omega)}_{\infty,0}(\mathbf{0}, \mathbf{0}; \beta, b, z, \epsilon)\big].$$

Afterwards consider the quantity:

$$\mathcal{Q}_0(\beta, b, z, \epsilon) := \frac{i}{2\pi} \int_{\Gamma_K} \mathrm{d}\xi\, (\xi - \xi_0)^2 \mathfrak{f}_\epsilon(\beta, z; \xi) \left( \frac{1}{|\Lambda_L|} \int_{\Lambda_L} \mathrm{d}\mathbf{x}\, \mathcal{I}_0(\mathbf{x}, \mathbf{x}; b, \omega, \xi) - \mathbb{E}\big[\mathcal{I}_0(\mathbf{0}, \mathbf{0}; b, \omega, \xi)\big] \right).$$

Then by using (4.2) there exists a constant $c = c(\beta_1, K) > 0$ s.t. $\forall\, \beta \in [\beta_1, \beta_2]$ and $\forall\, z \in K$:

$$|\mathcal{Q}_0(\beta, b, z, \epsilon)| \leq c \int_{\Gamma_K} |\mathrm{d}\xi|\, |\xi - \xi_0|^2 e^{-\beta_1 \Re \xi} \left| \frac{1}{|\Lambda_L|} \int_{\Lambda_L} \mathrm{d}\mathbf{x}\, \mathcal{I}_0(\mathbf{x}, \mathbf{x}; b, \omega, \xi) - \mathbb{E}\big[\mathcal{I}_0(\mathbf{0}, \mathbf{0}; b, \omega, \xi)\big] \right|.$$

In view of (6.5), this proves the proposition for $n = 0$.

Similarly we consider the case of $n = 1$ from (5.24). In view of Propositions 5.3, 5.5 and 5.6, $\mathbb{P}$-a.s., $\forall b \in \mathbb{R}$ and $\forall \xi \in \Gamma_K$, (6.5) also holds true if we consider now the kernel $\mathcal{I}_1(\mathbf{x}, \mathbf{x}; b, \omega, \xi)$ instead of $\mathcal{I}_0(\mathbf{x}, \mathbf{x}; b, \omega, \xi)$. Hence, following the proof for the case of $n = 0$ step by step, we conclude the proof for the case of $n = 1$. This proves *i*). *ii*) also follows by the same arguments but we disregard the $b$-dependence since we only treat the zero-field case.

6.4. **Proof of Proposition 6.2.** Let $L > 0$ and $\kappa > 0$. We use the decomposition:

$$\Lambda_L = (\Lambda_L \setminus \Lambda_\kappa) \cup \Lambda_\kappa \quad \text{with } \Lambda_\kappa := \{\mathbf{x} \in \Lambda_L \,:\, \mathrm{d}(\mathbf{x}) < \kappa\}, \tag{6.6}$$

where we set $\mathrm{d}(\mathbf{x}) := \mathrm{dist}(\mathbf{x}, \partial\Lambda_L)$. Note that the definition (6.6) implies:

$$|\Lambda_\kappa| = \mathcal{O}(L^2) \quad \text{when } L \to \infty. \tag{6.7}$$

Hereafter we denote by $\chi_{\Lambda_\kappa}$ the characteristic function of $\Lambda_\kappa$.

**Lemma 6.3.** *$\mathbb{P}$-a.s. on $\Omega$, $\forall b \in \mathbb{R}$, $\forall 0 < L < \infty$ and $\forall \xi \in \Gamma_K$, then:*
*i)* $\mathbf{x} \mapsto \big(R_L^{(1)}(\mathbf{x},\mathbf{y};b,\omega,\xi) - R_\infty^{(1)}(\mathbf{x},\mathbf{y};b,\omega,\xi)\big)\big|_{\mathbf{y}=\mathbf{x}}$ *and* $\mathbf{x} \mapsto \big((i\nabla_\mathbf{x} + b\mathbf{a}(\mathbf{x}))(R_L^{(1)}(\mathbf{x},\mathbf{y};b,\omega,\xi) - R_\infty^{(1)}(\mathbf{x},\mathbf{y};b,\omega,\xi))\big)\big|_{\mathbf{y}=\mathbf{x}}$ *are continuous functions on $\Lambda_L \setminus \Lambda_\kappa$.*
*ii) There exists a constant $\gamma > 0$ and a polynomial $p(\cdot)$ both $L$-independent s.t. on $\Lambda_L^2 \setminus D_L$:*

$$\begin{gathered}|R_L^{(1)}(\mathbf{x},\mathbf{y};b,\omega,\xi) - R_\infty^{(1)}(\mathbf{x},\mathbf{y};b,\omega,\xi)| \\ \leq |p(\xi)|(1+|\mathbf{x}|^\alpha+|\mathbf{y}|^\alpha)\mathrm{e}^{-\frac{\gamma}{1+|\xi|}|\mathbf{x}-\mathbf{y}|}\left(\frac{\chi_{\Lambda_\kappa}(\mathbf{x})}{|\mathbf{x}-\mathbf{y}|} + \frac{\chi_{\Lambda_\kappa}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} + \mathrm{e}^{-\frac{\gamma}{1+|\xi|}(\mathrm{d}(\mathbf{x})+\mathrm{d}(\mathbf{y}))}\right),\end{gathered} \tag{6.8}$$

$$\begin{gathered}|(i\nabla_\mathbf{x} + b\mathbf{a}(\mathbf{x}))(R_L^{(1)}(\mathbf{x},\mathbf{y};b,\omega,\xi) - R_\infty^{(1)}(\mathbf{x},\mathbf{y};b,\omega,\xi))| \\ \leq |p(\xi)|(1+|\mathbf{x}|^\alpha+|\mathbf{y}|^\alpha)^2\mathrm{e}^{-\frac{\gamma}{1+|\xi|}|\mathbf{x}-\mathbf{y}|}\left(\frac{\chi_{\Lambda_\kappa}(\mathbf{x})}{|\mathbf{x}-\mathbf{y}|^2} + \frac{\chi_{\Lambda_\kappa}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^2} + \mathrm{e}^{-\frac{\gamma}{1+|\xi|}(\mathrm{d}(\mathbf{x})+\mathrm{d}(\mathbf{y}))}\right).\end{gathered} \tag{6.9}$$

*Proof.* Under the conditions of Lemma 6.3 and from the Green's identity, we have on $\Lambda_L^2 \setminus D_L$:

$$R_L^{(1)}(\mathbf{x},\mathbf{y};b,\omega,\xi) - R_\infty^{(1)}(\mathbf{x},\mathbf{y};b,\omega,\xi) = -\frac{1}{2}\int_{\partial\Lambda_L} \mathrm{d}\sigma(\mathbf{z})\, R_\infty^{(1)}(\mathbf{x},\mathbf{z};b,\omega,\xi)[\mathbf{n_z}\cdot\nabla_\mathbf{z} R_L^{(1)}(\mathbf{z},\mathbf{y};b,\omega,\xi)],$$

where $\mathrm{d}\sigma(\mathbf{z})$ is the measure on $\partial\Lambda_L$ and $\mathbf{n_z}$ the outer normal to $\partial\Lambda_L$. From (2.9), Lemma 2.4 and since $\mathbf{z} \in \partial\Lambda_L$, then $\mathbb{P}$-a.s., $\forall b \in \mathbb{R}$ there exists a constant $\gamma > 0$ and a polynomial $p(\cdot)$ s.t. $\forall \xi \in \Gamma_K$ the integrand in the r.h.s. is a continuous function on $(\Lambda_L \setminus \Lambda_\kappa)^2$ satisfying

$$\begin{gathered}\int_{\partial\Lambda_L} \mathrm{d}\sigma(\mathbf{z}) \left| R_\infty^{(1)}(\mathbf{x},\mathbf{z};b,\omega,\xi)[\mathbf{n_z}\cdot\nabla_\mathbf{z} R_L^{(1)}(\mathbf{z},\mathbf{y};b,\omega,\xi)]\right| \\ \leq |p(\xi)|(1+|\mathbf{x}|^\alpha+|\mathbf{y}|^\alpha)\mathrm{e}^{-\frac{\gamma}{1+|\xi|}|\mathbf{x}-\mathbf{y}|}\mathrm{e}^{-\frac{\gamma}{1+|\xi|}(\mathrm{d}(\mathbf{x})+\mathrm{d}(\mathbf{y}))}.\end{gathered} \tag{6.10}$$

Then (6.10) together with (2.9) imply (6.8). We also have on $\Lambda_L^2 \setminus D_L$:

$$\begin{gathered}(i\nabla_\mathbf{x} + b\mathbf{a}(\mathbf{x}))(R_L^{(1)}(\mathbf{x},\mathbf{y};b,\omega,\xi) - R_\infty^{(1)}(\mathbf{x},\mathbf{y};b,\omega,\xi)) = \\ -\frac{1}{2}\int_{\partial\Lambda_L} \mathrm{d}\sigma(\mathbf{z})\,(i\nabla_\mathbf{x} + b\mathbf{a}(\mathbf{x}))R_\infty^{(1)}(\mathbf{x},\mathbf{z};b,\omega,\xi)[\mathbf{n_z}\cdot\nabla_\mathbf{z} R_L^{(1)}(\mathbf{z},\mathbf{y};b,\omega,\xi)],\end{gathered}$$

Then Lemma 2.4 together with the above arguments show that $(i\nabla_\mathbf{x}+b\mathbf{a}(\mathbf{x}))(R_L^{(1)}(\cdot\,,\cdot\,;b,\omega,\xi) - R_\infty^{(1)}(\cdot\,,\cdot\,;b,\omega,\xi))$ is also a continuous function on $(\Lambda_L \setminus \Lambda_\kappa)^2$ and (6.9) holds. ∎

**Remark 6.4.** Definitions (3.2) and (3.3) then imply the following estimates. Under the same conditions as in Lemma 6.3, $\forall (\mathbf{x},\mathbf{y}) \in \Lambda_L^2 \setminus D_L$ and for $j = 1, 2$:

$$\begin{gathered}|T_{j,L}(\mathbf{x},\mathbf{y};b,\omega,\xi) - T_{j,\infty}(\mathbf{x},\mathbf{y};b,\omega,\xi)| \\ \leq |p(\xi)|(1+|\mathbf{x}|^\alpha+|\mathbf{y}|^\alpha)^2\mathrm{e}^{-\frac{\gamma}{1+|\xi|}|\mathbf{x}-\mathbf{y}|}\left(\frac{\chi_{\Lambda_\kappa}(\mathbf{x})}{|\mathbf{x}-\mathbf{y}|} + \frac{\chi_{\Lambda_\kappa}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} + \mathrm{e}^{-\frac{\gamma}{1+|\xi|}(\mathrm{d}(\mathbf{x})+\mathrm{d}(\mathbf{y}))}\right),\end{gathered} \tag{6.11}$$

for another constant $\gamma > 0$ and polynomial $p(\cdot\,)$.

6.4.1. *The case of n=0.* Under the conditions of Proposition 6.2, we have:

$$\frac{1}{|\Lambda_L|}\frac{1}{\beta}\int_{\Lambda_L} \mathrm{d}\mathbf{x}\left(\mathcal{L}_L^{(\omega)}(\mathbf{x},\mathbf{x};\beta,b,z,\epsilon) - \mathcal{L}_{\infty,0}^{(\omega)}(\mathbf{x},\mathbf{x};\beta,b,z,\epsilon)\right) =$$
$$\frac{1}{\beta}\frac{1}{L^3}\frac{i}{2\pi}\Bigg(\int_{\Gamma_K}\mathrm{d}\xi\,\mathfrak{f}_\epsilon(\beta,z;\xi)\int_{\Lambda_L\setminus\Lambda_\kappa}\mathrm{d}\mathbf{x}\left(R_L^{(1)}(\mathbf{x},\mathbf{y};b,\omega,\xi) - R_\infty^{(1)}(\mathbf{x},\mathbf{y};b,\omega,\xi)\right)\big|_{\mathbf{y}=\mathbf{x}} +$$
$$\int_{\Gamma_K}\mathrm{d}\xi\,(\xi-\xi_0)\mathfrak{f}_\epsilon(\beta,z;\xi)\int_{\Lambda_\kappa}\mathrm{d}\mathbf{x}\,(R_L(b,\omega,\xi_0)R_L(b,\omega,\xi))(\mathbf{x},\mathbf{x}) - (R_\infty(b,\omega,\xi_0)R_\infty(b,\omega,\xi))(\mathbf{x},\mathbf{x})\Bigg).$$

Here we have used the first resolvent equation and the integral Cauchy formula to rewrite the second term of the r.h.s. of this expression. From Remark 2.2 and (4.2) this second term is bounded above by $c \times |\Lambda_\kappa|$ for some constant $c = c(\beta_1, K) > 0$. On the other hand, due to (6.8), a straightforward calculus leads to:

$$\left|\int_{\Lambda_L\setminus\Lambda_\kappa}\mathrm{d}\mathbf{x}\left(R_L^{(1)}(\mathbf{x},\mathbf{y};b,\omega,\xi) - R_\infty^{(1)}(\mathbf{x},\mathbf{y};b,\omega,\xi)\right)\big|_{\mathbf{y}=\mathbf{x}}\right| \leq |p(\xi)|L^{2+\alpha},$$

for another polynomial $p(\cdot\,)$. Hence as in the proof of Proposition 6.1, then $\mathbb{P}$-a.s., $\forall\beta \in [\beta_1,\beta_2]$, $\forall b \in \mathbb{R}$, $\forall z \in K$ and for $L$ sufficiently large:

$$\left|\frac{1}{|\Lambda_L|}\int_{\Lambda_L}\mathrm{d}\mathbf{x}\left(\mathcal{L}_L^{(\omega)}(\mathbf{x},\mathbf{x};\beta,b,z,\epsilon) - \mathcal{L}_{\infty,0}^{(\omega)}(\mathbf{x},\mathbf{x};\beta,b,z,\epsilon)\right)\right| \leq c\frac{1}{L^{1-\alpha}},$$

for some constant $c = c(\beta_1, b, K) > 0$. This proves the case of $n = 0$.

6.4.2. *The cases of n=1,2.* For any $b \in \mathbb{R}$, $L \in (0,\infty]$ and $\xi \in \Gamma_K$, introduce the notations:

$$\mathcal{K}_{L,j}(\mathbf{x},\mathbf{z}) = \mathcal{K}_{L,j}(\mathbf{x},\mathbf{z};b,\omega,\xi) := (-1)^{j+1}R_L^{(1)}(\mathbf{x},\mathbf{z};b,\omega,\xi)T_{j,L}(\mathbf{z},\mathbf{x};b,\omega,\xi) \quad j = 1,2,$$

where $(\mathbf{x},\mathbf{z}) \in \Lambda_L^2$ and $\mathbf{x} \neq \mathbf{z}$. Furthermore for $j = 1,2$ set:

$$u_{L,j}^{(\omega)}(b,\xi) := \int_{\Lambda_L}\mathrm{d}\mathbf{x}\int_{\Lambda_L}\mathrm{d}\mathbf{z}\,\mathcal{K}_{L,j}(\mathbf{x},\mathbf{z}), \quad u_{\infty,j}^{(\omega)}(b,\xi) := \int_{\Lambda_L}\mathrm{d}\mathbf{x}\int_{\Lambda_L}\mathrm{d}\mathbf{z}\,\mathcal{K}_{\infty,j}(\mathbf{x},\mathbf{z}), \tag{6.12}$$
$$v_{\infty,j}^{(\omega)}(b,\xi) := \int_{\Lambda_L}\mathrm{d}\mathbf{x}\int_{\mathbb{R}^3\setminus\Lambda_L}\mathrm{d}\mathbf{z}\,\mathcal{K}_{\infty,j}(\mathbf{x},\mathbf{z}).$$

Obviously we have the following estimate:

$$\left|\int_{\Lambda_L}\mathrm{d}\mathbf{x}\left(\int_{\Lambda_L}\mathrm{d}\mathbf{z}\,\mathcal{K}_{L,j}(\mathbf{x},\mathbf{z}) - \int_{\mathbb{R}^3}\mathrm{d}\mathbf{z}\,\mathcal{K}_{\infty,j}(\mathbf{x},\mathbf{z})\right)\right| \leq |u_{L,j}^{(\omega)}(b,\xi) - u_{\infty,j}^{(\omega)}(b,\xi)| + |v_{\infty,j}^{(\omega)}(b,\xi)|.$$

We want to estimate each term in the above r.h.s. Firstly let us prove the following result. $\mathbb{P}$-a.s., $\forall b \in \mathbb{R}$, there exists a polynomial $p(\cdot\,)$ s.t. $\forall\xi \in \Gamma_K$ and for large $L$:

$$|u_{L,j}^{(\omega)}(b,\xi) - u_{\infty,j}^{(\omega)}(b,\xi)| \leq |p(\xi)|L^{2+2\alpha}. \tag{6.13}$$

With our previous notations we have:

$$u_{L,j}^{(\omega)}(b,\xi) - u_{\infty,j}^{(\omega)}(b,\xi) = \int_{\Lambda_L}\mathrm{d}\mathbf{x}\int_{\Lambda_L}\mathrm{d}\mathbf{z}\left(\mathcal{K}_{L,j}(\mathbf{x},\mathbf{z}) - \mathcal{K}_{\infty,j}(\mathbf{x},\mathbf{z})\right). \tag{6.14}$$

By leaving out the dependence on $b$, $\omega$ and $\xi$ for the kernels, we have for $j = 1,2$:

$$|\mathcal{K}_{L,j}(\mathbf{x},\mathbf{z}) - \mathcal{K}_{\infty,j}(\mathbf{x},\mathbf{z})| \leq |(R_L^{(1)} - R_\infty^{(1)})(\mathbf{x},\mathbf{z})T_{j,L}(\mathbf{z},\mathbf{x})| + |R_\infty^{(1)}(\mathbf{x},\mathbf{z})(T_{j,L} - T_{j,\infty})(\mathbf{z},\mathbf{x})|. \tag{6.15}$$

In view of (6.15), let us estimate the r.h.s of (6.14). By (5.18)-(5.17) for $L=\infty$, (6.8) and (6.11), then $\mathbb{P}$-a.s., $\forall b\in\mathbb{R}$ there exists a constant $\gamma>0$ and a polynomial $p(\cdot)$ s.t. $\forall\xi\in\Gamma_K$, for $L$ large and $\forall(\mathbf{x},\mathbf{z})\in\Lambda_L^2\setminus D_L$:

$$|\mathcal{K}_{L,j}(\mathbf{x},\mathbf{z})-\mathcal{K}_{\infty,j}(\mathbf{x},\mathbf{z})|\leq|p(\xi)|L^{2\alpha}\Big(\frac{\chi_{\Lambda_\kappa}(\mathbf{x})}{|\mathbf{x}-\mathbf{z}|}+\frac{\chi_{\Lambda_\kappa}(\mathbf{z})}{|\mathbf{x}-\mathbf{z}|}+\mathrm{e}^{-\frac{\gamma}{1+|\xi|}(\mathrm{d}(\mathbf{z})+\mathrm{d}(\mathbf{x}))}\Big)\frac{\mathrm{e}^{-2\frac{\gamma}{1+|\xi|}|\mathbf{x}-\mathbf{z}|}}{|\mathbf{x}-\mathbf{z}|}. \quad (6.16)$$

Put the estimate (6.16) in the r.h.s. of (6.14) and a straightforward computation leads to (6.13). On the other hand from (5.18)-(5.17), we have:

$$\forall\,\xi\in\Gamma_K,\quad|\mathcal{K}_{\infty,j}(\mathbf{x},\mathbf{z})|\leq|p(\xi)|(1+|\mathbf{z}|^\alpha)\frac{\mathrm{e}^{-\frac{\gamma}{1+|\xi|}|\mathbf{x}-\mathbf{z}|}}{|\mathbf{x}-\mathbf{z}|^2},$$

for another $\gamma>0$ and polynomial $p(\cdot)$. We now use that there exists a constant $c>0$ s.t.:

$$\forall\gamma>0,\,\forall L\in(0,\infty),\quad\int_{\Lambda_L}\mathrm{d}\mathbf{x}\int_{\mathbb{R}^3\setminus\Lambda_L}\mathrm{d}\mathbf{z}\,\frac{\mathrm{e}^{-\gamma|\mathbf{x}-\mathbf{z}|}}{|\mathbf{x}-\mathbf{z}|^k}\leq c\gamma^{-(2+k)}L^2\quad k=1,2.$$

Then $\mathbb{P}$-a.s., $\forall b\in\mathbb{R}$, there exists another polynomial $p(\cdot)$ s.t. $\forall\xi\in\Gamma_K$ and for large $L$:

$$|v_{\infty,j}^{(\omega)}(b,\xi)|\leq|p(\xi)|L^{2+\alpha}. \quad (6.17)$$

Then (6.13)-(6.17) imply:

$$\left|\int_{\Lambda_L}\mathrm{d}\mathbf{x}\Big(\int_{\Lambda_L}\mathrm{d}\mathbf{z}\,\mathcal{K}_{L,j}(\mathbf{x},\mathbf{z})-\int_{\mathbb{R}^3}\mathrm{d}\mathbf{z}\,\mathcal{K}_{\infty,j}(\mathbf{x},\mathbf{z})\Big)\right|\leq|p(\xi)|L^{2+2\alpha}. \quad (6.18)$$

From this analysis we conclude the case of $n=1$. Indeed from (6.13)-(6.17), $\mathbb{P}$-a.s., $\forall b\in\mathbb{R}$ there exists a constant $c=c(\beta_1,K,b)>0$ s.t. $\forall\beta\in[\beta_1,\beta_2]$, $\forall z\in K$ and large $L$:

$$\begin{aligned}&\left|\frac{1}{|\Lambda_L|}\int_{\Lambda_L}\mathrm{d}\mathbf{x}\left(\frac{\partial\mathcal{L}_L^{(\omega)}}{\partial b}(\mathbf{x},\mathbf{x};\beta,b,z,\epsilon)-\mathcal{L}_{\infty,1}^{(\omega)}(\mathbf{x},\mathbf{x};\beta,b,z,\epsilon)\right)\right|\\&\quad\leq\frac{1}{|\Lambda_L|}\int_{\Gamma_K}|\mathrm{d}\xi||\mathfrak{f}_\epsilon(\beta,z;\xi)|\left|\int_{\Lambda_L}\mathrm{d}\mathbf{x}\left(\int_{\Lambda_L}\mathrm{d}\mathbf{z}\,\mathcal{K}_{L,1}(\mathbf{x},\mathbf{z})-\int_{\mathbb{R}^3}\mathrm{d}\mathbf{z}\,\mathcal{K}_{\infty,1}(\mathbf{x},\mathbf{z})\right)\right|\leq c\frac{1}{L^{1-2\alpha}}.\end{aligned}$$

We now complete the proof of the proposition. Define for any $b\in\mathbb{R}$, $L\in(0,\infty]$ and $\xi\in\Gamma_K$:

$$\mathcal{K}_{L,3}(\mathbf{x},\mathbf{Z}):=R_L^{(1)}(\mathbf{x},\mathbf{z}_1;b,\omega,\xi)T_{1,L}(\mathbf{z}_1,\mathbf{z}_2;b,\omega,\xi)T_{1,L}(\mathbf{z}_2,\mathbf{x};b,\omega,\xi),$$

where $\mathbf{Z}:=(\mathbf{z}_1,\mathbf{z}_2)$, $(\mathbf{x},\mathbf{z}_1,\mathbf{z}_2)\in\Lambda_L^3$ and $\mathbf{x}\neq\mathbf{z}_1\neq\mathbf{z}_2$. Introduce the quantities $u_{L,3}^{(\omega)}(b,\xi),u_{\infty,3}^{(\omega)}(b,\xi)$ as in (6.12) but with $\mathcal{K}_{L,3}(\mathbf{x},\mathbf{Z})$ instead of $\mathcal{K}_{L,j}(\mathbf{x},\mathbf{Z})$, $j=1,2$ and integrating w.r.t. the measure $\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{Z}$. Moreover set:

$$v_{\infty,3}^{(\omega)}(b,\xi):=\int_{\Lambda_L}\mathrm{d}\mathbf{x}\left\{\int_{\mathbb{R}^3\setminus\Lambda_L}\mathrm{d}\mathbf{z}_1\int_{\mathbb{R}^3}\mathrm{d}\mathbf{z}_2\,\mathcal{K}_{\infty,3}(\mathbf{x},\mathbf{Z})+\int_{\Lambda_L}\mathrm{d}\mathbf{z}_1\int_{\mathbb{R}^3\setminus\Lambda_L}\mathrm{d}\mathbf{z}_2\,\mathcal{K}_{\infty,3}(\mathbf{x},\mathbf{Z})\right\}. \quad (6.19)$$

Let us estimate $u_{L,3}^{(\omega)}(b,\xi)-u_{\infty,3}^{(\omega)}(b,\xi)$ with the same method as above. (6.15) is replaced with:

$$\begin{aligned}&|\mathcal{K}_{L,3}(\mathbf{x},\mathbf{Z})-\mathcal{K}_{\infty,3}(\mathbf{x},\mathbf{Z})|\leq|(R_L^{(1)}-R_\infty^{(1)})(\mathbf{x},\mathbf{z}_1)T_{1,L}(\mathbf{z}_1,\mathbf{z}_2)T_{1,L}(\mathbf{z}_2,\mathbf{x})|+\\&|R_\infty^{(1)}(\mathbf{x},\mathbf{z}_1)(T_{1,L}-T_{1,\infty})(\mathbf{z}_1,\mathbf{z}_2)T_{1,L}(\mathbf{z}_2,\mathbf{x})|+|R_\infty^{(1)}(\mathbf{x},\mathbf{z}_1)T_{1,\infty}(\mathbf{z}_1,\mathbf{z}_2)(T_{1,L}-T_{1,\infty})(\mathbf{z}_2,\mathbf{x})|.\end{aligned}$$

Here it is convenient to set $\mathbf{z}_0 = \mathbf{z}_3 = \mathbf{x}$. By using (5.18), (5.17) together with (6.8), (6.9), then $\mathbb{P}$-a.s., $\forall b \in \mathbb{R}$ there exists a constant $\gamma > 0$ and a polynomial $p(\cdot)$ s.t. $\forall \xi \in \Gamma_K$, for large $L$ and $\forall (\mathbf{x}, \mathbf{Z}) \in \Lambda_L^3$ with $\mathbf{x} \neq \mathbf{z}_1 \neq \mathbf{z}_2$:

$$|\mathcal{K}_{L,3}(\mathbf{x}, \mathbf{Z}) - \mathcal{K}_{\infty,3}(\mathbf{x}, \mathbf{Z})| \leq |p(\xi)| L^{3\alpha} \mathrm{e}^{-\frac{\gamma}{1+|\xi|} \sum_{l=0}^{2} |\mathbf{z}_l - \mathbf{z}_{l+1}|} \times \\ \times \Big(\prod_{l=0}^{2} \frac{1}{|\mathbf{z}_{l+1} - \mathbf{z}_l|}\Big) \sum_{l=0}^{2} (\chi_{\Lambda_\kappa}(\mathbf{z}_l) + |\mathbf{z}_{l+1} - \mathbf{z}_l| \mathrm{e}^{-\frac{\gamma}{1+|\xi|}(\mathrm{d}(\mathbf{z}_{l+1}) + \mathrm{d}(\mathbf{z}_l))}). \quad (6.20)$$

So from (6.20) by a tedious computation, we obtain that $\mathbb{P}$-a.s., $\forall b \in \mathbb{R}$, there exists another polynomial $p(\cdot)$ s.t. $\forall \xi \in \Gamma_K$ and for large $L$:

$$|u_{L,3}^{(\omega)}(b, \xi) - u_{\infty,3}^{(\omega)}(b, \xi)| \leq |p(\xi)| L^{2+3\alpha}. \quad (6.21)$$

We also have the estimate:

$$|\mathcal{K}_{\infty,3}(\mathbf{x}, \mathbf{Z})| \leq |p(\xi)|(1 + |\mathbf{z}_1|^\alpha)(1 + |\mathbf{z}_2|^\alpha)\Big(\prod_{l=0}^{2} \frac{1}{|\mathbf{z}_{l+1} - \mathbf{z}_l|}\Big) \mathrm{e}^{-\frac{\gamma}{2(1+|\xi|)} \sum_{l=0}^{2} |\mathbf{z}_l - \mathbf{z}_{l+1}|},$$

for another constant $\gamma > 0$ and polynomial $p(\cdot)$. In view of (6.19), by some straightforward estimates, $\mathbb{P}$-a.s., $\forall b \in \mathbb{R}$, there exists another polynomial $p(\cdot)$ s.t. $\forall\, \xi \in \Gamma_K$ and large $L$:

$$|v_{\infty,3}^{(\omega)}(b, \xi)| \leq |p(\xi)| L^{2+2\alpha}. \quad (6.22)$$

Consequently,

$$\left| \int_{\Lambda_L} \mathrm{d}\mathbf{x} \left( \int_{\Lambda_L^2} \mathrm{d}\mathbf{Z}\, \mathcal{K}_{L,3}(\mathbf{x}, \mathbf{Z}) - \int_{\mathbb{R}^3} \mathrm{d}\mathbf{Z}\, \mathcal{K}_{\infty,3}(\mathbf{x}, \mathbf{Z}) \right) \right| \leq |p(\xi)| L^{2+3\alpha}. \quad (6.23)$$

Let us prove the case $n = 2$. From (6.18) and (6.23), $\mathbb{P}$-a.s., $\forall b \in \mathbb{R}$ there exists a constant $c = c(\beta_1, K, b) > 0$ s.t. $\forall \beta \in [\beta_1, \beta_2]$, $\forall z \in K$ and for large $L$:

$$\left| \frac{1}{|\Lambda_L|} \int_{\Lambda_L} \mathrm{d}\mathbf{x} \left( \frac{\partial^2 \mathcal{L}_L^{(\omega)}}{\partial b^2}(\mathbf{x}, \mathbf{x}; \beta, b, z, \epsilon) - \mathcal{L}_{\infty,2}^{(\omega)}(\mathbf{x}, \mathbf{x}; \beta, b, z, \epsilon) \right) \right| \\ \leq \frac{1}{|\Lambda_L|} \int_{\Gamma_K} |\mathrm{d}\xi| |\mathfrak{f}_\epsilon(\beta, z; \xi)| \left\{ \left| \int_{\Lambda_L} \mathrm{d}\mathbf{x} \left( \int_{\Lambda_L^2} \mathrm{d}\mathbf{Z}\, \mathcal{K}_{L,3}(\mathbf{x}, \mathbf{Z}) - \int_{\mathbb{R}^6} \mathrm{d}\mathbf{Z}\, \mathcal{K}_{\infty,3}(\mathbf{x}, \mathbf{Z}) \right) \right| + \\ \left| \int_{\Lambda_L} \mathrm{d}\mathbf{x} \left( \int_{\Lambda_L} \mathrm{d}\mathbf{z}\, \mathcal{K}_{L,2}(\mathbf{x}, \mathbf{z}) - \int_{\mathbb{R}^3} \mathrm{d}\mathbf{z}\, \mathcal{K}_{\infty,2}(\mathbf{x}, \mathbf{z}) \right) \right| \right\} \leq c \frac{1}{L^{1-3\alpha}}. \quad (6.24)$$

Since we suppose that $0 < \alpha < \frac{1}{3}$, the proposition follows. ■

**Remark 6.5.** (6.18), (6.23) and (4.5) show that the finite-volume magnetization and susceptibility, if they are defined, are uniformly bounded in $L$ provided that $\alpha \in (0, \frac{1}{3})$. In fact these considerations can be extended to estimate $\mathcal{X}_{L,3}^{(\omega)}(\beta, b, z, \epsilon)$ in (4.15). Indeed let:

$$\sum_{j=1}^{2} \left( \int_{\Lambda_L} \mathrm{d}\mathbf{x}\, \mathcal{U}_{L,j}^{(\omega)}(\mathbf{x}, \mathbf{x}; b, \xi) - \int_{\Lambda_L} \mathrm{d}\mathbf{x}\, \mathcal{U}_{\infty,j}^{(\omega)}(\mathbf{x}, \mathbf{x}; b, \xi) \right), \quad (6.25)$$

where $\mathcal{U}_{L,j}^{(\omega)}(\cdot\,, \cdot\,; b, \xi)$ are defined in (4.16) and $\mathcal{U}_{\infty,j}^{(\omega)}(\cdot\,, \cdot\,; b, \xi)$ are defined in the same way but with $L = \infty$. (6.25) can be estimated as above. In that case, handling heavy technicalities, we can show that $\mathbb{P}$-a.s., $\forall b \in \mathbb{R}$ there exists a polynomial $p(\cdot)$ s.t. $\forall \xi \in \Gamma_K$, the quantity in

(6.25) is bounded above by $|p(\xi)|L^{2+4\alpha}$. It follows that if $\alpha \in (0, \frac{1}{4})$, then (4.15) and (4.2) imply that $\mathbb{P}$-a.s., $\forall \beta > 0$, $\forall b \in \mathbb{R}$ and $\forall z \in K$, $\mathcal{X}_{L,3}^{(\omega)}(\beta, b, z, \epsilon)$ is also bounded uniformly in $L \in (0, \infty)$.

## 7. Appendix

7.1. **Proof of Lemma 2.4.** For all $L \in (0, \infty]$ and $b \in \mathbb{R}$ denote by $G_L(\cdot\,,\cdot\,; t, b)$ the integral kernel of the strongly continuous semigroup $\{\mathrm{e}^{-tH_{0,L}(b)}, t > 0\}$; here $H_{0,L}(b), L < \infty, L = \infty$ is the free operator defined respectively in (1.5) and (1.6) with $V^{(\omega)} = 0$. It is known that $G_L(\cdot\,,\cdot\,; t, b)$ is smooth and obeys, see [4, Eq. (2.31) & (4.13)]:

$$\forall (\mathbf{x}, \mathbf{y}) \in \Lambda_L^2, \quad |(i\nabla_{\mathbf{x}} + b\mathbf{a}(\mathbf{x}))G_L(\mathbf{x}, \mathbf{y}; t, b)| \leq c(1 + |b|)^3(1 + t)^5 t^{-2} \mathrm{e}^{-\frac{|\mathbf{x}-\mathbf{y}|^2}{16t}}, \tag{7.1}$$

where $c > 0$ is a $L$-independent constant. Let $\lambda < \min\{0, E_0\}$. By using the Laplace transform [15, Eq. (2.5)] in the kernels sense, we get on $\Lambda_L^2 \setminus D_L$:

$$(i\nabla_{\mathbf{x}} + b\mathbf{a}(\mathbf{x}))R_{0,L}^{(1)}(\mathbf{x}, \mathbf{y}; b, \lambda) = \int_0^{+\infty} \mathrm{d}t\, \mathrm{e}^{\lambda t}(i\nabla_{\mathbf{x}} + b\mathbf{a}(\mathbf{x}))G_L(\mathbf{x}, \mathbf{y}; t, b), \tag{7.2}$$

where $R_{0,L}^{(1)}(\cdot\,,\cdot\,; b, \lambda)$ denotes the integral kernel of $R_{0,L}(b, \lambda) := (H_{0,L}(b) - \lambda)^{-1}$, $L \in (0, \infty]$. Due to (7.1), (7.2) is well-defined and the function $(\mathbf{x}, \mathbf{y}) \mapsto (i\nabla_{\mathbf{x}} + b\mathbf{a}(\mathbf{x}))R_{0,L}^{(1)}(\mathbf{x}, \mathbf{y}; b, \lambda)$ is jointly continuous on $\Lambda_L^2 \setminus D_L$. Besides, there exists a constant $c = c(\lambda) > 0$ s.t.:

$$\forall (\mathbf{x}, \mathbf{y}) \in \Lambda_L^2 \setminus D_L, \quad |(i\nabla_{\mathbf{x}} + b\mathbf{a}(\mathbf{x}))R_{0,L}^{(1)}(\mathbf{x}, \mathbf{y}; b, \lambda)| \leq c(1 + |b|)^3 \frac{\mathrm{e}^{-\frac{\sqrt{-\lambda}}{2\sqrt{2}}|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x} - \mathbf{y}|^2}. \tag{7.3}$$

Now $\mathbb{P}$-a.s., $\forall b \in \mathbb{R}$, $\forall \lambda < \min\{0, E_0\}$ and $\forall L \in (0, \infty]$, the second resolvent equation:

$$R_L(b, \omega, \lambda) = R_{0,L}(b, \lambda) - R_{0,L}(b, \lambda)V^{(\omega)}R_L(b, \omega, \lambda), \tag{7.4}$$

holds in the bounded operators sense. Indeed, $\mathbb{P}$-a.s., $\forall b \in \mathbb{R}$, $V^{(\omega)}R_L(b, \omega, \lambda)$ is bounded if $L < \infty$. When $L = \infty$, $R_{0,\infty}(b, \lambda)V^{(\omega)}R_\infty(b, \omega, \lambda)$ is defined as the closure of the same operator defined on the domain $(H_\infty(b, \omega) - \lambda)\mathcal{C}_0^\infty(\mathbb{R}^3)$ which is dense in $L^2(\mathbb{R}^3)$. Hence, we have

$$(i\nabla_{\mathbf{x}} + b\mathbf{a}(\mathbf{x}))R_L^{(1)}(\mathbf{x}, \mathbf{y}; b, \omega, \lambda) = (i\nabla_{\mathbf{x}} + b\mathbf{a}(\mathbf{x}))R_{0,L}^{(1)}(\mathbf{x}, \mathbf{y}; b, \lambda) - \mathcal{M}_L(\mathbf{x}, \mathbf{y}; b, \omega, \lambda), \tag{7.5}$$

$$\mathcal{M}_L(\mathbf{x}, \mathbf{y}; b, \omega, \lambda) := \int_{\Lambda_L} \mathrm{d}\mathbf{z}\, (i\nabla_{\mathbf{x}} + b\mathbf{a}(\mathbf{x}))R_{0,L}^{(1)}(\mathbf{x}, \mathbf{z}; b, \lambda)V^{(\omega)}(\mathbf{z})R_L^{(1)}(\mathbf{z}, \mathbf{y}; b, \omega, \lambda).$$

According to the decomposition $V^{(\omega)} = V_1^{(\omega)} + V_2^{(\omega)}$, set on $\Lambda_L^2 \setminus D_L$:

$$\mathcal{M}_L(\mathbf{x}, \mathbf{y}; b, \omega, \lambda) = \mathcal{M}_{L,1}(\mathbf{x}, \mathbf{y}; b, \omega, \lambda) + \mathcal{M}_{L,2}(\mathbf{x}, \mathbf{y}; b, \omega, \lambda).$$

From (2.9), (7.3), Lemma 7.4 with (R1) if $l = 1$ and Lemma 7.1 *i)* with (1.3) if $l = 2$, together with (7.2) and (7.4), then $\mathbb{P}$-a.s., $\forall b \in \mathbb{R}$, $\forall \lambda < \min\{0, E_0\}$, $\forall L \in (0, \infty]$ and $\forall (\mathbf{x}, \mathbf{y}) \in \Lambda_L^2 \setminus D_L$, $(i\nabla_{\mathbf{x}} + b\mathbf{a}(\mathbf{x}))R_{0,L}^{(1)}(\mathbf{x}, \cdot\,; b, \lambda)V_l^{(\omega)}(\cdot\,)R_L^{(1)}(\cdot\,, \mathbf{y}; b, \omega, \lambda) \in L^1(\Lambda_L)$ $\forall (\mathbf{x}, \mathbf{y}) \in \Lambda_L^2 \setminus D_L$. From Lemmas 7.1 and 7.4 then $(\mathbf{x}, \mathbf{y}) \mapsto \mathcal{M}_{L,l}(\mathbf{x}, \mathbf{y}; b, \omega, \lambda)$ are jointly continuous on $\Lambda_L^2 \setminus D_L$. Therefore this also holds for $(\mathbf{x}, \mathbf{y}) \mapsto (i\nabla_{\mathbf{x}} + b\mathbf{a}(\mathbf{x}))R_L^{(1)}(\mathbf{x}, \mathbf{y}; b, \omega, \lambda)$.
On the other hand, from Lemmas 7.2 and 7.3 together with (2.9), (7.3) and the inequality:

$$|\mathbf{x} - \mathbf{z}|^{-1}|\mathbf{z} - \mathbf{y}|^{-1} \leq |\mathbf{x} - \mathbf{y}|^{-1}(|\mathbf{x} - \mathbf{z}|^{-1} + |\mathbf{z} - \mathbf{y}|^{-1}) \quad \mathbf{x} \neq \mathbf{y} \neq \mathbf{z}, \tag{7.6}$$

then there exists a constant $c > 0$ s.t.:

$$\forall (\mathbf{x},\mathbf{y}) \in \Lambda_L^2 \setminus D_L, \quad |\mathcal{M}_{L,1}(\mathbf{x},\mathbf{y};b,\omega,\lambda)| \leq c(1+|b|)^3 \frac{\mathrm{e}^{-c^{'}(\lambda)|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|}, \tag{7.7}$$

where $c^{'}(\lambda) := \frac{1}{2}\min\{\frac{\sqrt{-\lambda}}{2\sqrt{2}}, \frac{\gamma}{1+|\lambda|}\} = \frac{1}{2}\frac{\gamma}{1+|\lambda|}$ if $\lambda < 0$ is chosen large enough. From (1.3), (2.9), (7.3) together with (7.6), then there exists $c > 0$ s.t.:

$$\forall (\mathbf{x},\mathbf{y}) \in \Lambda_L^2 \setminus D_L, \quad |\mathcal{M}_{L,2}(\mathbf{x},\mathbf{y};b,\omega,\lambda)| \leq c(1+|b|)^3 (1+|\mathbf{x}|^\alpha + |\mathbf{y}|^\alpha) \frac{\mathrm{e}^{-\frac{c^{'}(\lambda)}{4}|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|}. \tag{7.8}$$

Therefore in view of (7.5), then (7.3) together with (7.7) and (7.8) imply that $\mathbb{P}$-a.s., $\forall \lambda < \min\{0, E_0\}$, there exists $c > 0$ s.t. $\forall L \in (0,\infty]$, $\forall b \in \mathbb{R}$, we have on $\Lambda_L^2 \setminus D_L$:

$$|(i\nabla_\mathbf{x} + b\mathbf{a}(\mathbf{x}))R_L^{(1)}(\mathbf{x},\mathbf{y};b,\omega,\lambda)| \leq c(1+|b|)^3(1+|\mathbf{x}|^\alpha + |\mathbf{y}|^\alpha)\frac{\mathrm{e}^{-\frac{c^{'}(\lambda)}{8}|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|^2}. \tag{7.9}$$

Now the first resolvent equation allows us to write $\forall (\mathbf{x},\mathbf{y}) \in \Lambda_L^2 \setminus D_L$:

$$\begin{aligned} &(i\nabla_\mathbf{x} + b\mathbf{a}(\mathbf{x}))R_L^{(1)}(\mathbf{x},\mathbf{y};b,\omega,\xi) = (i\nabla_\mathbf{x} + b\mathbf{a}(\mathbf{x}))R_L^{(1)}(\mathbf{x},\mathbf{y};b,\omega,\lambda) + \mathcal{N}_L(\mathbf{x},\mathbf{y};b,\omega,\xi,\lambda), \\ &\mathcal{N}_L(\mathbf{x},\mathbf{y};b,\omega,\xi,\lambda) := (\xi-\lambda)\int_{\Lambda_L} \mathrm{d}\mathbf{z}\,(i\nabla_\mathbf{x} + b\mathbf{a}(\mathbf{x}))R_L^{(1)}(\mathbf{x},\mathbf{z};b,\omega,\lambda)R_L^{(1)}(\mathbf{z},\mathbf{y};b,\omega,\xi). \end{aligned} \tag{7.10}$$

Due to (2.9), (7.9) and Lemma 7.1 *i)*, $\mathbb{P}$-a.s., $\forall b \in \mathbb{R}$, $\forall L \in (0,\infty]$, $\forall \eta > 0$, $\forall \lambda < \min\{0, E_0\}$, and $\forall \xi \in \mathbb{C}$ s.t. $d(\xi) \geq \eta$, $(\mathbf{x},\mathbf{y}) \mapsto \mathcal{N}_L(\mathbf{x},\mathbf{y};b,\omega,\xi,\lambda)$ is jointly continuous on $\Lambda_L^2 \setminus D_L$. This proves Lemma 2.4 *i)*. *ii)* follows from (2.9), (7.9) and the fact that for $|\lambda|$ large enough, there exists $\gamma' > 0$ s.t. $\forall \xi \in \mathbb{C}$, $d(\xi) \geq \eta$, $\min\{\frac{c^{'}(\lambda)}{16}, \frac{\gamma}{1+|\xi|}\} \geq \frac{\gamma'}{1+|\xi|}$. ■

7.2. **Some kernel estimates.** Here we give some useful estimates needed in this paper.

**Lemma 7.1.** *Let $U \subseteq \mathbb{R}^3$ be an open set and $D := \{(\mathbf{x},\mathbf{y}) \in U^2 : \mathbf{x} = \mathbf{y}\}$.*
*Let $K_l(\cdot\,,\cdot) : U^2 \setminus D \to \mathbb{C}$, $l = 1,2$ be integral kernels satisfying:*
*(h1) $K_1(\cdot\,,\mathbf{z})$ and $K_2(\mathbf{z},\cdot)$ are continuous on $U \setminus \{\mathbf{z}\}$ for almost all $\mathbf{z} \in U$.*
*(h2) There exist real numbers $c_l, \gamma_l > 0$ and $\nu_l, \mu_l \geq 0$ as well as $\delta_l \in [0,3)$ s.t.:*

$$\forall (\mathbf{x},\mathbf{y}) \in U^2 \setminus D, \quad |K_l(\mathbf{x},\mathbf{y})| \leq c_l(|\mathbf{x}|^{\nu_l} + |\mathbf{y}|^{\mu_l})\frac{\mathrm{e}^{-\gamma_l|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|^{\delta_l}} \quad l = 1,2. \tag{7.11}$$

*Then:*
*i) $K_1(\mathbf{x},\cdot\,)K_2(\cdot\,,\mathbf{y}) \in L^1(U)$ for all $(\mathbf{x},\mathbf{y}) \in U^2 \setminus D$. Furthermore the map:*
*$(\mathbf{x},\mathbf{y}) \mapsto \mathcal{K}(\mathbf{x},\mathbf{y}) := \int_U \mathrm{d}\mathbf{z}\, K_1(\mathbf{x},\mathbf{z})K_2(\mathbf{z},\mathbf{y})$ is jointly continuous on $U^2 \setminus D$.*
*ii) Under the additional assumption $\delta_1 + \delta_2 \in [0,3)$, $K_1(\mathbf{x},\cdot\,)K_2(\cdot\,,\mathbf{x}) \in L^1(U)$ for all $\mathbf{x} \in U$. Moreover $\mathbf{x} \mapsto \mathcal{K}(\mathbf{x},\mathbf{x})$ is continuous on $U$.*

This result is obtained by using standard arguments, see e.g. [9, Sect. 3].

**Lemma 7.2.** *Consider the assumptions (h1)-(h2) of Lemma 7.1 but with $\mu_l, \nu_l = 0$.*
*i) Let $\delta_l = 1,2$ and $\gamma := \min\{\gamma_1,\gamma_2\}$. Then there exists a constant $c > 0$ s.t. on $U^2$:*

$$\int_U \mathrm{d}\mathbf{z}\, |K_1(\mathbf{x},\mathbf{z})K_2(\mathbf{z},\mathbf{y})| \leq \frac{c}{\gamma}\mathrm{e}^{-\frac{\gamma}{2}|\mathbf{x}-\mathbf{y}|} \times \begin{cases} \frac{1}{|\mathbf{x}-\mathbf{y}|^{\min\{\delta_1,\delta_2\}}} & \text{if } \delta_1,\delta_2 \neq 1 \text{ and } \mathbf{x} \neq \mathbf{y} \\ 1 & \text{if } \delta_1,\delta_2 = 1 \end{cases}. \tag{7.12}$$

*ii) Let $\gamma_l = \gamma$ and $\delta_l = l$, $l = 1, 2$. Then $\forall k \geq 2$ there exists a constant $c > 0$ s.t. on $U^2$:*

$$\int_{U^k} \mathrm{d}\mathbf{z}_1 \cdots \mathrm{d}\mathbf{z}_k \, |K_1(\mathbf{x}, \mathbf{z}_1) K_1(\mathbf{z}_1, \mathbf{z}_2) \cdots K_1(\mathbf{z}_k, \mathbf{y})| \leq \frac{c}{\gamma^{2k-1}} \mathrm{e}^{-\frac{\gamma}{2^k}|\mathbf{x}-\mathbf{y}|}, \tag{7.13}$$

$$\int_{U^k} \mathrm{d}\mathbf{z}_1 \cdots \mathrm{d}\mathbf{z}_k \, |K_1(\mathbf{x}, \mathbf{z}_1) K_2(\mathbf{z}_1, \mathbf{z}_2) \cdots K_2(\mathbf{z}_k, \mathbf{y})| \leq \frac{c}{\gamma^k} \frac{\mathrm{e}^{-\frac{\gamma}{2^k}|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|} \quad \text{if } \mathbf{x} \neq \mathbf{y}. \tag{7.14}$$

*Proof.* The main ingredient is the following estimate. For all $\gamma > 0$ and $\delta \in [0, 3)$,

$$\sup_{\mathbf{x} \in U} \int_{\mathbb{R}^3} \mathrm{d}\mathbf{y} \, \frac{\mathrm{e}^{-\gamma|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|^\delta} = \int_0^\infty \mathrm{d}r \, r^{2-\delta} \mathrm{e}^{-\gamma r} = \frac{\Gamma(3-\delta)}{\gamma^{3-\delta}}, \tag{7.15}$$

where $\Gamma(\cdot)$ denotes the usual Gamma Euler function. When $\delta_1 = \delta_2 = 1$, we have:

$$\forall (\mathbf{x}, \mathbf{y}) \in U^2, \quad \int_U \mathrm{d}\mathbf{z} \, |K_1(\mathbf{x}, \mathbf{z}) K_2(\mathbf{z}, \mathbf{y})| \leq c_1 c_2 \mathrm{e}^{-\frac{\gamma}{2}|\mathbf{x}-\mathbf{y}|} \int_{\mathbb{R}^3} \mathrm{d}\mathbf{z} \, \frac{\mathrm{e}^{-\frac{\gamma_1}{2}|\mathbf{x}-\mathbf{z}|}}{|\mathbf{x}-\mathbf{z}|} \frac{\mathrm{e}^{-\frac{\gamma_2}{2}|\mathbf{z}-\mathbf{y}|}}{|\mathbf{z}-\mathbf{y}|}.$$

Then by the Cauchy-Schwarz inequality and (7.15), (7.12) follows. Similarly we get the cases of $\delta_1 = 2, \delta_2 = 1$ and $\delta_1 = \delta_2 = 2$ from (7.6) combined with (7.15).
The estimates (7.13) and (7.14) are obtained by induction from the above arguments. ∎

**Lemma 7.3.** *Let $\gamma > 0$, $\delta \in (0, 3)$ and $p > \frac{3}{(3-\delta)}$. Suppose that $V \in L^p_{\mathrm{uloc}}(\mathbb{R}^3)$. Then $\forall \mathbf{x} \in \mathbb{R}^3$, $V(\cdot)\mathrm{e}^{-\gamma|\mathbf{x}-\cdot|}|\mathbf{x}-\cdot|^{-\delta} \in L^1(\mathbb{R}^3)$, and there exists a constant $c = c(\gamma, \delta, \|V\|_{p,\mathrm{uloc}}) > 0$ s.t.:*

$$\sup_{\mathbf{x} \in \mathbb{R}^3} \|V(\cdot)\mathrm{e}^{-\gamma|\mathbf{x}-\cdot|}|\mathbf{x}-\cdot|^{-\delta}\|_1 \leq c.$$

*Proof.* Let $\frac{1}{p} + \frac{1}{q} = 1$ and $p > \frac{3}{(3-\delta)}$. By the Hölder inequality, we get:

$$\int_{|\mathbf{x}-\mathbf{y}|<1} \mathrm{d}\mathbf{y} \, |V(\mathbf{y})| \frac{\mathrm{e}^{-\gamma|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|^\delta} \leq \|V\|_{p,\mathrm{uloc}} \left( \int_{|\mathbf{x}-\mathbf{y}|<1} \mathrm{d}\mathbf{y} \, \frac{\mathrm{e}^{-q\gamma|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|^{q\delta}} \right)^{\frac{1}{q}} < \infty.$$

On the other hand, with same $p, q$, we have:

$$\int_{|\mathbf{x}-\mathbf{y}|\geq 1} \mathrm{d}\mathbf{y} \, |V(\mathbf{y})| \frac{\mathrm{e}^{-\gamma|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|^\delta} \leq \left( \sum_{k=1}^\infty \int_{k<|\mathbf{x}-\mathbf{y}|\leq k+1} \mathrm{d}\mathbf{y} \, |V(\mathbf{y})|^p \mathrm{e}^{-\frac{p\gamma}{2}|\mathbf{x}-\mathbf{y}|} \right)^{\frac{1}{p}} \left( \int_{|\mathbf{x}-\mathbf{y}|\geq 1} \mathrm{d}\mathbf{y} \, \mathrm{e}^{-\frac{q\gamma}{2}|\mathbf{x}-\mathbf{y}|} \right)^{\frac{1}{q}}.$$

Since the domain $k < |\mathbf{x}-\mathbf{y}| \leq k+1$ is covered by *cste* $\times k^2$ unit balls, then the above r.h.s. is bounded from above by $c\|V\|_{p,\mathrm{uloc}}\big(\sum_{k=1}^\infty k^2 \mathrm{e}^{-\frac{p\gamma}{2}k}\big)^{\frac{1}{p}} < \infty$ for some constant $c > 0$. ∎

From Lemma 7.1 together with Lemma 7.3, we finally prove:

**Lemma 7.4.** *Consider the assumptions (h1)-(h2) of Lemma 7.1. Let $V \in L^p_{\mathrm{uloc}}(\mathbb{R}^3)$ with $p > \frac{3}{3-\max\{\delta_1,\delta_2\}}$ if $\delta_1 + \delta_2 \neq 0$, elsewhere $p \geq 1$. Then $K_1(\mathbf{x}, \cdot)V(\cdot)K_2(\cdot, \mathbf{y}) \in L^1(U)$ for all $\mathbf{x} \neq \mathbf{y}$. Furthermore, $(\mathbf{x}, \mathbf{y}) \mapsto \int_U \mathrm{d}\mathbf{z} \, K_1(\mathbf{x}, \mathbf{z})V(\mathbf{z})K_2(\mathbf{z}, \mathbf{y})$ is jointly continuous on $U^2 \setminus D$.*

*Proof.* Consider only the 'most tricky' case which occurs when $U = \mathbb{R}^3$. Note first that the estimate in $(h2)$ can be rewritten on $U^2 \setminus D$ as:

$$|K_1(\mathbf{x}, \mathbf{y})| \leq c_1(1 + |\mathbf{x}|^{\nu_1+\mu_1}) \frac{\mathrm{e}^{-\frac{\gamma_1}{2}|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|^{\delta_1}}, \quad |K_2(\mathbf{x}, \mathbf{y})| \leq c_2(1 + |\mathbf{y}|^{\nu_2+\mu_2}) \frac{\mathrm{e}^{-\frac{\gamma_2}{2}|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|^{\delta_2}}, \tag{7.16}$$

for another constants $c_1, c_2 > 0$. Set $J(\mathbf{x}, \mathbf{y}; \cdot) := K_1(\mathbf{x}, \cdot)V(\cdot)K_2(\cdot, \mathbf{y})$. Let $0 < \varsigma < \frac{1}{2}|\mathbf{x} - \mathbf{y}|$ and denote by $\mathcal{B}(\cdot, \varsigma)$ the open ball having the radius $\varsigma > 0$. From (7.16), there exists a constant $c = c(|\mathbf{x}|, |\mathbf{y}|) > 0$ s.t. $\forall \mathbf{z} \in \mathcal{B}(\mathbf{x}, \varsigma)$, $|J(\mathbf{x}, \mathbf{y}; \mathbf{z})| \leq c|V(\mathbf{z})|\mathrm{e}^{-\frac{\gamma_1}{2}|\mathbf{x}-\mathbf{z}|}|\mathbf{x} - \mathbf{z}|^{-\delta_1}$. Then by Lemma 7.3, $J(\mathbf{x}, \mathbf{y}; \cdot\,) \in L^1(\mathcal{B}(\mathbf{x}, \varsigma))$ as soon as $p > 3/(3 - \delta_1)$. On the same way $J(\mathbf{x}, \mathbf{y}; \cdot\,) \in L^1(\mathcal{B}(\mathbf{y}, \varsigma))$ as soon as $p > 3/(3 - \delta_2)$. Besides there exists a constant $c^{'} = c^{'}(|\mathbf{x}|, |\mathbf{y}|) > 0$ s.t. $\forall \mathbf{z} \in \mathbb{R}^3 \setminus (\mathcal{B}(\mathbf{x}, \varsigma) \cup \mathcal{B}(\mathbf{y}, \varsigma))$, $|J(\mathbf{x}, \mathbf{y}; \mathbf{z})| \leq c\mathrm{e}^{-\frac{\gamma_1}{2}|\mathbf{x}-\mathbf{z}|}|V(\mathbf{z})|\mathrm{e}^{-\frac{\gamma_2}{2}|\mathbf{z}-\mathbf{y}|}$. As $\sup_{\mathbf{x}\in\mathbb{R}^3} \|V(\cdot\,)\mathrm{e}^{-\frac{\gamma_1}{2}|\mathbf{x}-\cdot\,|}\|_p, \sup_{\mathbf{y}\in\mathbb{R}^3} \|\mathrm{e}^{-\frac{\gamma_2}{2}|\cdot\,-\mathbf{y}|}\|_q < \infty$ whenever $p, q \geq 1$, then by the Hölder inequality, $J(\mathbf{x}, \mathbf{y}; \cdot\,) \in L^1(\mathbb{R}^3 \setminus (\mathcal{B}(\mathbf{x}, \varsigma) \cup \mathcal{B}(\mathbf{y}, \varsigma)))$. Therefore $J(\mathbf{x}, \mathbf{y}; \cdot\,) \in L^1(\mathbb{R}^3)$ provided that $p > 3/(3 - \max(\delta_1, \delta_2))$. By standard arguments, the continuity property follows. ■

## 8. Acknowledgments

The authors thank Horia Cornean for fruitful discussions about this problem and his kind invitations to department of mathematics of Aalborg university, where a part of this work was done.

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